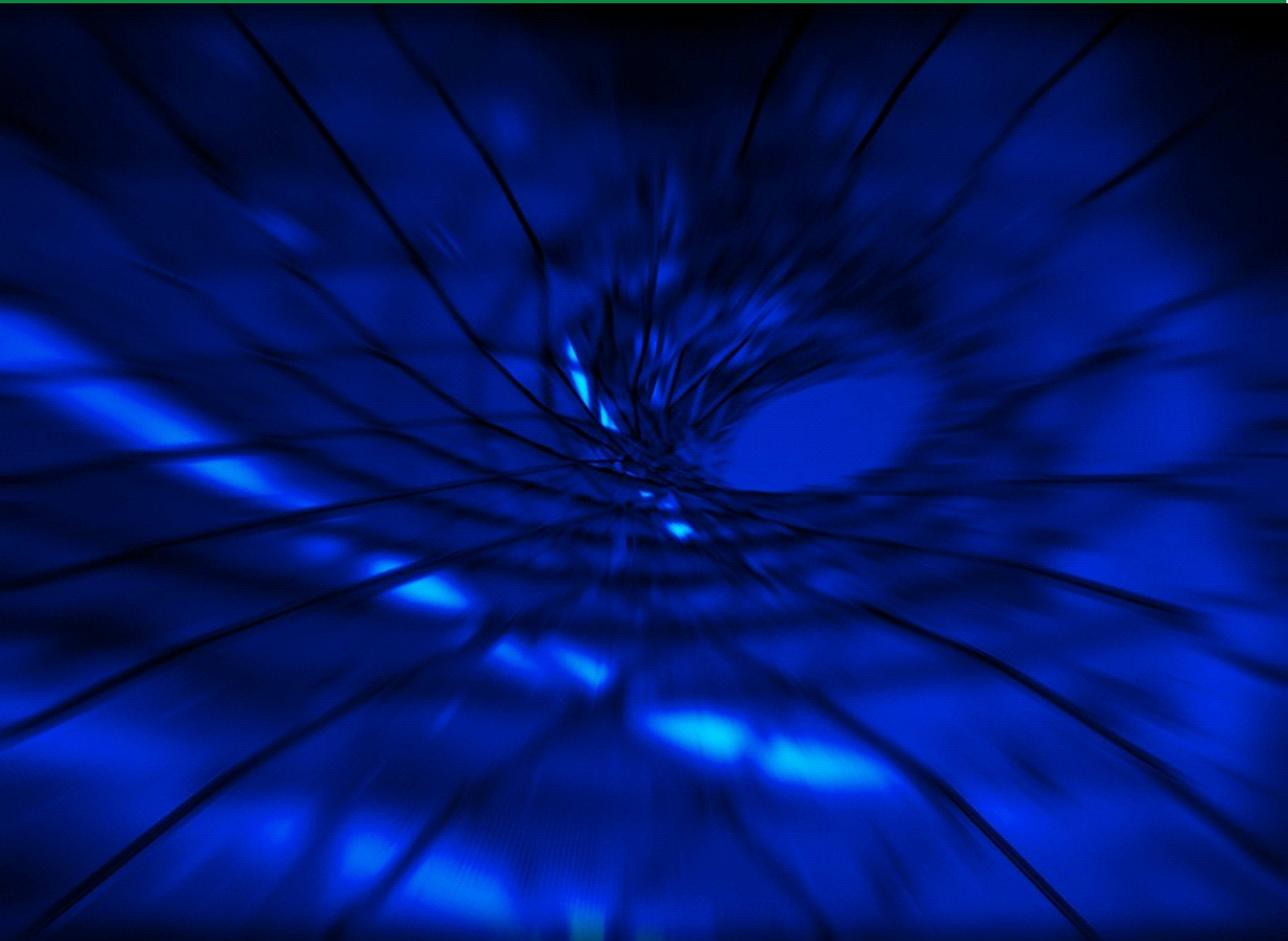


LEIF MEJLBRO

**TOPOLOGICAL AND METRIC SPACES,
BANACH SPACES...**

**...AND BOUNDED OPERATORS - FUNCTIONAL ANALYSIS
EXAMPLES C-2**



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Leif Mejlbro

Topological and Metric Spaces,
Banach Spaces and Bounded Operators

Topological and Metric Spaces, Banach Spaces and Bounded Operators

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Introduction

This is the second volume containing examples from FUNCTIONAL ANALYSIS. The topics here are limited to *Topological and metric spaces*, *Banach spaces* and *Bounded operators*.

Unfortunately errors cannot be avoided in a first edition of a work of this type. However, the author has tried to put them on a minimum, hoping that the reader will meet with sympathy the errors which do occur in the text.

Leif Mejlbro
24th November 2009

1 Topological and metric spaces

1.1 Weierstraß's approximation theorem

Example 1.1 Let $\varphi \in C^1([0, 1])$. It follows from Weierstraß's approximation theorem that $B_{n,\varphi}(\theta)$ converges uniformly towards $\varphi(\theta)$ and that $B_{n,\varphi'}(\theta)$ converges uniformly towards $\varphi'(\theta)$ on $[0, 1]$.

Prove that $B'_{n,\varphi}(\theta) \rightarrow \varphi'(\theta)$ uniformly on $[0, 1]$.

HINT: First prove that $B'_{n,\varphi}(\theta) - B_{n-1,\varphi'}(\theta)$ converges uniformly towards 0 on $[0, 1]$.

Next prove that if $\varphi \in C^\infty([0, 1])$, then we have for every $k \in \mathbb{N}$ that $B_{n,\varphi}^{(k)}(\theta) \rightarrow \varphi^{(k)}(\theta)$ uniformly on $[0, 1]$.

NOTATION. We use here the notation

$$B_{n,\varphi}(\theta) = \sum_{k=0}^n \varphi\left(\frac{k}{n}\right) \cdot \binom{n}{k} \cdot \theta^k (1-\theta)^{n-k}$$

for the so-called *Bernstein polynomials*. \diamond

First write

$$\begin{aligned} B'_{n,\varphi}(\theta) - B_{n-1,\varphi'}(\theta) &= \sum_{k=0}^n \varphi\left(\frac{k}{n}\right) \cdot \binom{n}{k} \cdot \frac{d}{d\theta} \{\theta^k (1-\theta)^{n-k}\} \\ &\quad - \sum_{k=0}^{n-1} \varphi'\left(\frac{k}{n-1}\right) \cdot \binom{n-1}{k} \cdot \theta^k (1-\theta)^{n-1-k}. \end{aligned}$$

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Here

$$\frac{d}{d\theta}\{\theta^k(1-\theta)^{n-k}\} = \begin{cases} n\theta^{n-1}, & \text{for } k = n, \\ k \cdot \theta^{k-1}(1-\theta)^{n-k} - (n-k)\theta^k(1-\theta)^{n-1-k}, & \text{for } 0 < k < n, \\ -n(1-\theta)^{n-1}, & \text{for } k = 0. \end{cases}$$

For $0 < k < n$ we perform the calculation

$$\begin{aligned} \binom{n}{k} \frac{d}{d\theta}\{\theta^k(1-\theta)^{n-k}\} &= \frac{n!}{k!(n-k)!} \{k\theta^{k-1}(1-\theta)^{n-k} - (n-k)\theta^k(1-\theta)^{n-1-k}\} \\ &= \frac{n!}{(k-1)!(n-k)!} \theta^{k-1}(1-\theta)^{n-k} - \frac{n!}{k!(n-k-1)!} \theta^k(1-\theta)^{n-1-k} \\ &= n \binom{n-1}{k-1} \theta^{k-1}(1-\theta)^{n-k} - n \binom{n-1}{k} \theta^k(1-\theta)^{n-1-k}. \end{aligned}$$

Hence

$$\begin{aligned} B'_{n,\varphi}(\theta) &= \sum_{k=0}^n \varphi\left(\frac{k}{n}\right) \cdot \binom{n}{k} \cdot \frac{d}{d\theta}\{\theta^k(1-\theta)^{n-k}\} \\ &= \varphi(0) \cdot \{-n(1-\theta)^{n-1}\} + \varphi(1) \cdot n\theta^{n-1} + n \sum_{k=1}^{n-1} \varphi\left(\frac{k}{n}\right) \cdot \binom{n-1}{k-1} \theta^{k-1}(1-\theta)^{n-k} \\ &\quad - n \sum_{k=1}^{n-1} \varphi\left(\frac{k}{n}\right) \cdot \binom{n-1}{k} \theta^k(1-\theta)^{n-1-k} \\ &= n \{\varphi(1) \cdot \theta^{n-1} - \varphi(0) \cdot (1-\theta)^{n-1}\} + n \sum_{k=0}^{n-2} \varphi\left(\frac{k+1}{n}\right) \cdot \binom{n-1}{k} \theta^k(1-\theta)^{n-1-k} \\ &\quad - n \sum_{k=1}^{n-1} \varphi\left(\frac{k}{n}\right) \cdot \binom{n-1}{k} \theta^k(1-\theta)^{n-1-k} \\ &= n \sum_{k=0}^{n-1} \left\{ \varphi\left(\frac{k+1}{n}\right) - \varphi\left(\frac{k}{n}\right) \right\} \cdot \binom{n-1}{k} \theta^k(1-\theta)^{n-1-k} \\ &= \sum_{k=0}^{n-1} \frac{\varphi\left(\frac{k+1}{n}\right) - \varphi\left(\frac{k}{n}\right)}{\frac{1}{n}} \cdot \binom{n-1}{k} \theta^k(1-\theta)^{n-1-k}. \end{aligned}$$

Whence by insertion,

$$B'_{n,\varphi}(\theta) - B_{n-1,\varphi'}(\theta) = \sum_{k=0}^{n-1} \left\{ \frac{\varphi\left(\frac{k+1}{n}\right) - \varphi\left(\frac{k}{n}\right)}{\frac{1}{n}} - \varphi'\left(\frac{k}{n-1}\right) \right\} \cdot \binom{n-1}{k} \theta^k(1-\theta)^{n-1-k}.$$

We have assumed from the beginning that $\varphi \in C^1([0,1])$, thus

$$\frac{\varphi\left(\frac{k+1}{n}\right) - \varphi\left(\frac{k}{n}\right)}{\frac{1}{n}} - \varphi'\left(\frac{k}{n-1}\right) = \frac{1}{n} \varepsilon\left(\frac{1}{n}\right)$$

uniformly, so the remainder term is estimated uniformly independently of k . In fact, it follows from the Mean Value Theorem that

$$\frac{\varphi\left(\frac{k+1}{n}\right) - \varphi\left(\frac{k}{n}\right)}{\frac{1}{n}} = \varphi'(\xi), \quad \text{for et passende } \xi \in \left] \frac{k}{n} \frac{k+1}{n} \left[,$$

and as $\frac{k}{n} - \frac{k}{n-1} = -\frac{k}{n(n-1)}$, we get

$$\left| \frac{k}{n} - \frac{k}{n-1} \right| \leq \frac{1}{n-1},$$

and since φ' is continuous,

$$\varphi'\left(\frac{k}{n}\right) - \varphi'\left(\frac{k}{n-1}\right) \rightarrow 0 \quad \text{ligeligt.}$$

From this follows precisely that

$$\frac{\varphi\left(\frac{k+1}{n}\right) - \varphi\left(\frac{k}{n}\right)}{\frac{1}{n}} - \varphi'\left(\frac{k}{n-1}\right) = \varphi'\left(\frac{k}{n}\right) - \varphi'\left(\frac{k}{n-1}\right) \frac{1}{n} \varepsilon \left(\frac{1}{n}\right)$$

uniformly, and the claim is proved.

Finally, we get by induction that if $\varphi \in C^k([0, 1])$, then $B_{n,\varphi}^{(k)}(\theta) \rightarrow \varphi^{(k)}(\theta)$ uniformly on $[0, 1]$.

Example 1.2 Let φ be a real continuous function defined for $x \geq 0$, and assume that $\lim_{x \rightarrow \infty} \varphi(x)$ exists (and is finite). Show that for $\varepsilon > 0$ there are $n \in \mathbb{N}$ and constants $a_k, k = 0, 1, \dots, n$, such that

$$\left| \varphi(x) - \sum_{k=0}^n a_k e^{-kx} \right| \leq \varepsilon$$

for all $x \geq 0$.

First note that the range of $e^{-x}, x \in [0, \infty[$, is $]0, 1]$, so we have $t = e^{-x} \in]0, 1]$, thus $x = \ln \frac{1}{t}$. The function $\psi(t)$, given by

$$\psi(t) = \begin{cases} \varphi\left(\ln \frac{1}{t}\right) & \text{for } t \in]0, 1], \\ \lim_{x \rightarrow \infty} \varphi(x) & \text{for } t = 0, \end{cases}$$

is continuous for $t \in [0, 1]$. It follows from Weierstraß's approximation theorem that there exists a polynomial $\sum_{k=0}^n a_k t^k$, such that

$$\left| \psi(t) - \sum_{k=0}^n a_k t^k \right| \leq \varepsilon \quad \text{for alle } t \in [0, 1].$$

Since $\varphi(x) = \psi(e^{-x})$ for $x \in [0, +\infty[$, we conclude that

$$\left| \varphi(x) - \sum_{k=0}^n a_k e^{-kx} \right| \leq \varepsilon \quad \text{for every } x \in [0, +\infty[.$$

1.2 Topological and metric spaces

Example 1.3 Let (M, d) be a metric space.

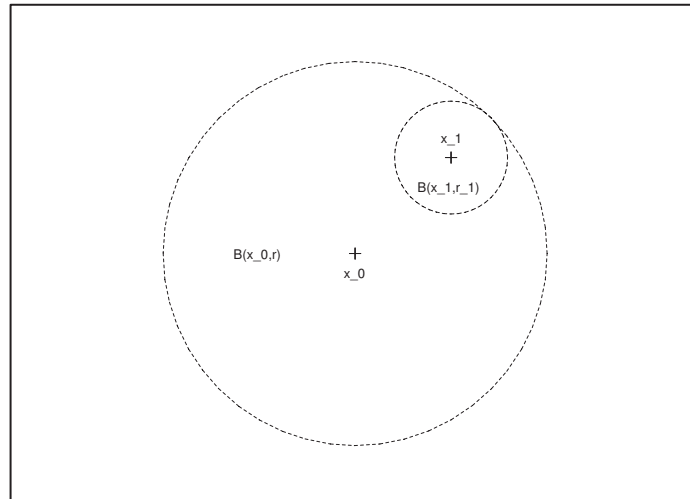
We define the open ball with centre x_0 and radius $r > 0$ by

$$B(x_0, r) = \{x \in M \mid d(x, x_0) < r\}.$$

We denote a subset $A \subset M$ open, if there for any $x_0 \in A$ is an open ball with centre x_0 contained in A .

Show that an open ball is an open set.

Show that the open sets defined in this way is a topology on M .



Let $x_1 \in B(x_0, r)$, i.e. $d(x_0, x_1) < r$. Choose

$$r_1 = r - d(x_0, x_1) > 0.$$

We claim that

$$B(x_1, r_1) \subseteq B(x_0, r).$$

If $x \in B(x_1, r_1)$, then

$$d(x_1, x) < r_1 = r - d(x_0, x_1),$$

and it follows by the triangle inequality that

$$d(x_0, x) \leq d(x_0, x_1) + d(x_1, x) < d(x_0, x_1) + r - d(x_0, x_1) = r,$$

proving that $x \in B(x_0, r)$. This holds for every $x \in B(x_1, r_1)$, so we have proved with the chosen radius r_1 that

$$B(x_1, r_1) \subseteq B(x_0, r),$$

hence every open ball is in fact an open set.

Then we shall prove that the system \mathcal{T} generated by all open balls is a topology. Thus a set $T \in \mathcal{T}$ is characterized by the property that for every $x \in T$ there exists an $r > 0$, such that $B(x, r) \subseteq T$.

- 1) It is trivial that M itself is an open set.
That \emptyset is open follows from the formal definition:

$$\forall x_0 \in \emptyset \exists r \in \mathbb{R}_+ : B(x_0, r) \subseteq \emptyset.$$

Since there is no point in \emptyset , the condition is trivially fulfilled.

- 2) Let $T = \bigcup_{j \in J} T_j$, where all $T_j \in \mathcal{T}$. If $x_0 \in T$, then there exists a $j \in J$, such that $x_0 \in T_j$. Since $T_j \in \mathcal{T}$, there exists an $r \in \mathbb{R}_+$, such that

$$B(x_0, r) \subseteq T_j \subseteq T,$$

thus $T \in \mathcal{T}$.

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- 3) Let $T = \bigcap_{j=1}^n T_j$, where all $T_j \in \mathcal{T}$. If $T = \emptyset$, there is nothing to prove. Therefore, let $x_0 \in T$. Then x_0 must lie in every $T_j \in \mathcal{T}$, $j = 1, \dots, n$, so there are constants $r_j \in \mathbb{R}_+$, $j = 1, \dots, n$, such that $B(x_0, r_j) \subseteq T_j$. Now put $t = \min r_j \in \mathbb{R}_+$ (notice that there is only a finite number of $r_j > 0$). Then

$$B(x_0, r) \subseteq B(x_0, r_j) \subseteq T_j \quad \text{for every } j = 1, \dots, n,$$

and hence also in the intersection,

$$B(x_0, r) \subseteq \bigcap_{j=1}^n T_j = T.$$

Using the definition of \mathcal{T} this means that $T \in \mathcal{T}$.

We have proved that \mathcal{T} is a topology.

Example 1.4 Let (M, d) be a metric space. We say that a mapping $T : M \rightarrow M$ is continuous in $x_0 \in M$ if, for any $\varepsilon > 0$ there is a $\delta > 0$ such that for all $x \in M$ we have

$$d(x_0, x) < \delta \implies d(Tx_0, Tx) < \varepsilon.$$

Show that T is continuous in x_0 if and only if

$$x_n \rightarrow x_0 \implies Tx_n \rightarrow Tx_0.$$

Show that T is continuous if the open sets are defined as in EXAMPLE 1.3.

Recall that $x_n \rightarrow x_0$ means that

$$(1) \quad \forall \delta \in \mathbb{R}_+ \exists n_0 \in \mathbb{N} \forall n \geq n_0 : d(x_n, x_0) < \delta.$$

Assume that T is continuous in $x_0 \in M$ and that $x_n \rightarrow x_0$. We shall prove that $Tx_n \rightarrow Tx_0$, i.e.

$$\forall \varepsilon \in \mathbb{R}_+ \exists n_0 \in \mathbb{N} \forall n \geq n_0 : d(Tx_n, Tx_0) < \varepsilon.$$

Let $\varepsilon \in \mathbb{R}_+$ be arbitrary. Since T is continuous, we can find to this $\varepsilon > 0$ a constant $\delta = \delta(\varepsilon) \in \mathbb{R}_+$, such that

$$(2) \quad \forall x \in M : d(x_0, x) < \delta \implies d(Tx_0, Tx) < \varepsilon.$$

Using that $x_n \rightarrow x_0$, we get by (1) an $n_0 \in \mathbb{N}$ corresponding to $\delta = \delta(\varepsilon)$ [in fact an $n_0 \in \mathbb{N}$ corresponding to $\varepsilon \in \mathbb{R}_+$], such that

$$\forall n \geq n_0 : d(x_n, x_0) < \delta = \delta(\varepsilon).$$

It follows from the continuity condition (2) that $d(Tx_0, Tx_n) < \varepsilon$ for $n \geq n_0$, hence

$$\forall \varepsilon \in \mathbb{R}_+ \exists n_0 \in \mathbb{N} \forall n \geq n_0 : d(Tx_n, Tx_0) < \varepsilon,$$

and we have proved that if T is continuous in $x_0 \in M$, then

$$x_n \rightarrow x_0 \implies Tx_n \rightarrow Tx_0.$$

Then assume that T is *not* continuous at $x_0 \in M$, thus

$$(3) \exists \varepsilon \in \mathbb{R}_+ \forall \delta \in \mathbb{R}_+ \exists x \in M : d(x_0, x) < \delta \wedge d(Tx_0, Tx) \geq \varepsilon.$$

We shall prove that there exists a sequence (x_n) , such that $x_n \rightarrow x_0$, while Tx_n does not converge towards Tx_0 .

Choose $\varepsilon > 0$ as in (3). Putting $\delta = \frac{1}{n}$ we get

$$\forall n \in \mathbb{N} \exists x_n \in M : d(x_0, x_n) < \frac{1}{n} \wedge d(Tx_0, Tx_n) \geq \varepsilon.$$

Then it follows that $x_n \rightarrow x_0$ and Tx_n cannot be arbitrarily close to Tx_0 , thus (Tx_n) does not converge towards Tx_0 .

Assume that $T^{\circ-1}(A)$ is open for every open set A . Choose $x_0 \in M$ and $A = B(Tx_0, \varepsilon)$. Then A is open, so $T^{\circ-1}(A)$ is open according to the assumption. It follows from $x_0 \in T^{\circ-1}(A)$ that there is a $\delta \in \mathbb{R}_+$, such that

$$B(x_0, \delta) \subseteq T^{\circ-1}(A).$$

For every $x_0 \in B(x_0, \delta)$, thus $d(x, x_0) < \delta$, we get $Tx \in B(Tx_0, \varepsilon)$, hence $d(Tx, Tx_0) < \varepsilon$, and we have proved that T is continuous.

Conversely, assume that T is continuous, and let A be an open set, thus

$$\forall x_0 \in A \exists r \in \mathbb{R}_+ : d(x_0, x) < r \implies x \in A.$$

We shall prove that $T^{\circ-1}(A)$ is open, i.e.

$$\forall y_0 \in T^{\circ-1}(A) \exists R \in \mathbb{R}_+ : B(y_0, R) \subseteq T^{\circ-1}(A).$$

This is done INDIRECTLY. *Assumem* that

$$\exists y_0 \in T^{\circ-1}(A) \forall R \in \mathbb{R}_+ : B(y_0, R) \setminus T^{\circ-1}(A) \neq \emptyset,$$

thus

$$\exists y_0 \in T^{\circ-1}(A) \forall R \in \mathbb{R}_+ \exists y \notin T^{\circ-1}(A) : d(y_0, y) < R.$$

Since T is continuous at y_0 , it follows that

$$\forall r \in \mathbb{R}_+ \exists R \in \mathbb{R}_+ \forall y \in M : d(y_0, y) < R \implies d(Ty_0, Ty) = d(x_0, Ty) < r.$$

We conclude that $Ty \in A$ contradicting that $y \notin T^{\circ-1}(A)$, and the claim is proved.

Example 1.5 In a set M is given a function d' from $M \times M$ to \mathbb{R} that satisfies

$$d'(x, y) = 0 \quad \text{if and only if} \quad x = y,$$

$$d'(x, y) \leq d'(z, x) + d'(z, y) \quad \text{for all } x, y, z \in M.$$

Show that (M, d') is a metric space.

If we choose $z = y$ in the latter assumption and then use the former one, we get

$$d'(x, y) \leq d'(y, x) + d'(y, y) = d'(y, x) + 0 = d'(y, x),$$

proving that

$$d'(x, y) \leq d'(y, x) \quad \text{for all } x, y \in M.$$

By interchanging x and y we obtain the opposite inequality, $d'(y, x) \leq d'(x, y)$, hence

$$d'(x, y) = d'(y, x) \quad \text{for all } x, y \in M,$$

and d' is symmetric.

Using this result on the latter assumption we get the triangle inequality

$$d'(x, y) \leq d'(x, z) + d'(z, y).$$

It only remains to prove that $d'(x, y) \geq 0$ for all $x, y \in M$ in order to conclude that d' is a metric. This follows from

$$0 = d'(x, x) \leq d'(x, y) + d'(y, x) = 2d'(x, y),$$

so the two conditions of the example suffice for d' being a metric.

Example 1.6 Let (M, d) be a metric space.

The diameter of a non-empty subset A of M is defined as

$$\delta(A) = \sup_{x, y \in A} d(x, y) \quad (\leq \infty).$$

Show that $\delta(A) = 0$ if and only if A contains only one point.

If $A = \{x\}$ only contains one point, then

$$\delta(A) = \sup_{x, y \in A} d(x, y) = d(x, x) = 0.$$

If A contains at least two points, choose $x, y \in A$, where $x \neq y$, from which we conclude that

$$\delta(A) = \sup_{t, z \in A} d(t, z) \geq d(x, y) > 0,$$

and the claim is proved.

Example 1.7 Let (M, d) be a metric space. Show that d_1 given by

$$d_1(x, y) = \frac{d(x, y)}{1 + d(x, y)} \quad \text{for } x, y \in M$$

is a metric on M .

Show that

$$\delta_1(A) = \sup_{x, y \in A} d_1(x, y) \leq 1$$

for all $A \subset M$.

Is it possible to find a subset A with $\delta_1(A) = 1$?


Show that $d_1(x_n, x) \rightarrow 0$ if and only if $d(x_n, x) \rightarrow 0$.

1) We shall first prove that

$$d_1(x, y) = \frac{d(x, y)}{1 + d(x, y)}, \quad x, y \in M,$$

is a metric.

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- a) It is trivial that $d_1(x, y) \geq 0$, because $d(x, y) \geq 0$.
 b) Then we see that $d_1(x, y) = 0$, if and only if the numerator $d(x, y) = 0$, i.e. if and only if $x = y$.
 c) The condition $d_1(x, y) = d_1(y, x)$ follows immediately from $d(x, y) = d(y, x)$.
 d) It remains only to prove the triangle inequality

$$d_1(x, y) \leq d_1(x, z) + d_1(z, y).$$

Now $d(x, y) \leq d(x, z) + d(z, y)$, and the function

$$f(t) = \frac{t}{1+t} = 1 - \frac{1}{1+t}, \quad t \geq 0,$$

is increasing. Hence

$$\begin{aligned} d_1(x, y) &= \frac{d(x, y)}{1+d(x, y)} = f(d(x, y)) \\ &\leq f(d(x, z) + d(z, y)) = \frac{d(x, z) + d(z, y)}{1+d(x, z) + d(z, y)} \\ &= \frac{d(x, z)}{1+d(x, z) + d(z, y)} + \frac{d(z, y)}{1+d(x, z) + d(z, y)} \\ &\leq \frac{d(x, z)}{1+d(x, z)} + \frac{d(z, y)}{1+d(z, y)} \\ &= d_1(x, z) + d_1(z, y). \end{aligned}$$

Summing up, we have proved that $d_1(x, y)$ is a metric on M .

- 2) It follows from

$$d_1(x, y) = \frac{d(x, y)}{1+d(x, y)} = 1 - \frac{1}{1+d(x, y)} \leq 1,$$

that

$$\delta_1(A) = \sup_{x, y \in A} d_1(x, y) \leq 1$$

for every subset A .

- 3) a) If the metric d is not bounded on M , then there are subsets A , such that $\delta_1(A) = 1$.
 In fact, we choose to every $n \in \mathbb{N}$ points $x_n, y_n \in M$, such that

$$d(x_n, y_n) \geq n - 1 \quad \text{for } n \in \mathbb{N}.$$

As mentioned previously, $f(t) = \frac{t}{1+t}$ is increasing, so

$$d_1(x_n, y_n) = f(d(x, y)) \geq f(n - 1) = \frac{n - 1}{n} = 1 - \frac{1}{n}.$$

Putting

$$A = \{x_n \mid n \in \mathbb{N}\} \cup \{y_n \mid n \in \mathbb{N}\},$$

it follows that $\delta_1(A) \geq 1 - \frac{1}{n}$ for every $n \in \mathbb{N}$, thus $\delta_1(A) \geq 1$. On the other hand, we have already proved that $\delta_1(A) \leq 1$, so we conclude that $\delta_1(A) = 1$.

b) If instead d is bounded on M , then M has itself a finite d -diameter, $\delta(M) = c < \infty$, and

$$\delta_1(M) = \frac{c}{1+c} = 1 - \frac{1}{1+c} < 1.$$

There are many examples of such metrics. The most obvious one is the well-known

$$d_0(x, y) = \begin{cases} 0 & \text{for } x = y, \\ 1 & \text{for } x \neq y, \end{cases}$$

where

$$\tilde{d}_0(x, y) = \begin{cases} 0 & \text{for } x = y, \\ \frac{1}{2} & \text{for } x \neq y. \end{cases}$$

We get another example by starting with the bounded d_1 above. Then

$$d_2(x, y) = \frac{d_1(x, y)}{1 + d_1(x, y)} = \frac{d(x, y)}{1 + 2d(x, y)},$$

with $\delta_2(A) \leq \frac{1}{2}$ for every subset $A \subseteq M$.

4) It follows from

$$d_1(x_n, x) = 1 - \frac{1}{1 + d(x_n, x)},$$

that the condition $d_1(x_n, x) \rightarrow 0$ is equivalent with $1 + d(x_n, x) \rightarrow 1$, thus with $d(x_n, x) \rightarrow 0$, and the claim is proved.

Example 1.8 Let (M_1, d_1) and (M_2, d_2) be metric spaces.

Show that $M_1 \times M_2$ can be made into a metric space by the following definition of a metric d :

$$d((x_1, x_2), (y_1, y_2)) = d_1(x_1, y_1) + d_2(x_2, y_2).$$

Show that also d^* given by

$$d^*((x_1, x_2), (y_1, y_2)) = \max\{d_1(x_1, y_1), d_2(x_2, y_2)\}$$

defines a metric on $M_1 \times M_2$.

1) Clearly,

$$d((x_1, x_2), (y_1, y_2)) \geq 0 \quad \text{and} \quad d^*((x_1, x_2), (y_1, y_2)) \geq 0.$$

2) If $(x_1, x_2) = (y_1, y_2)$, i.e. $x_1 = y_1$ and $x_2 = y_2$, then

$$d((x_1, x_2), (y_1, y_2)) = 0 \quad \text{and} \quad d^*((x_1, x_2), (y_1, y_2)) = 0.$$

Conversely, if

$$d((x_1, x_2), (y_1, y_2)) = 0 \quad \text{or} \quad d^*((x_1, x_2), (y_1, y_2)) = 0,$$

then both

$$d_1(x_1, y_1) = 0 \quad \text{and} \quad d_2(x_2, y_2) = 0,$$

and it follows that $x_1 = y_1$ and $x_2 = y_2$, and hence $(x_1, x_2) = (y_1, y_2)$.

- 3) The symmetry is obvious.
- 4) The triangle inequality holds for both d_1 and d_2 . Hence, it also holds for d and d^* . In fact,

$$\begin{aligned} d((x_1, x_2), (y_1, y_2)) &= d_1(x_1, y_1) + d_2(x_2, y_2) \\ &\leq \{d_1(x_1, z_1) + d_1(z_1, y_1)\} + \{d_2(x_2, z_2) + d_2(z_2, y_2)\} \\ &= \{d_1(x_1, z_1) + d_2(x_2, z_2)\} + \{d_1(z_1, y_1) + d_2(z_2, y_2)\} \\ &= d((x_1, x_2), (z_1, z_2)) + d((z_1, z_2), (y_1, y_2)), \end{aligned}$$

and

$$\begin{aligned} d^*((x_1, x_2), (y_1, y_2)) &= \max\{d_1(x_1, y_1), d_2(x_2, y_2)\} \\ &\leq \max\{d_1(x_1, z_1) + d_1(z_1, y_1), d_2(x_2, z_2) + d_2(z_2, y_2)\} \\ &\leq \max\{d_1(x_1, z_1), d_2(x_2, z_2)\} + \max\{d_1(z_1, y_1), d_2(z_2, y_2)\} \\ &= d^*((x_1, x_2), (z_1, z_2)) + d^*((z_1, z_2), (y_1, y_2)). \end{aligned}$$

Example 1.9 Show that in any set M we can define a metric by

$$d(x, y) = \begin{cases} 0 & \text{if } x = y, \\ 1 & \text{if } x \neq y. \end{cases}$$

Then we call (M, d) for a discrete metric space.

Characterize the sequences in M where $d(x_n, x) \rightarrow 0$.

- 1) Clearly, $d(x, y) \geq 0$.
- 2) Clearly, $d(x, y) = 0$, if and only if $x = y$.
- 3) Clearly, $d(x, y) = d(y, x)$.
- 4) Finally, it is almost trivial that

$$d(x, y) \leq d(x, z) + d(z, y),$$

because the left hand side is always ≤ 1 . If the right hand side is < 1 , then both $d(x, z) = 0$ and $d(z, y) = 0$, and we infer that $x = z$ and $z = y$, hence also $x = y$. This implies that the left hand side $d(x, y) = 0$, and the triangle inequality is fulfilled.

Summing up we have proved that (M, d) is a metric space.

If $d(x_n, x) \rightarrow 0$, then choose $\varepsilon = \frac{1}{2}$. There exists an $n_0 \in \mathbb{N}$, such that

$$d(x_n, x) < \varepsilon = \frac{1}{2} \quad \text{for } n \geq n_0.$$

This is only possible, if $d(x_n, x) = 0$, i.e. if

$$x_n = x \quad \text{for all } n \geq n_0.$$

We conclude that all the convergent sequences are constant eventually.

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Example 1.10 Let (M, d) be a metric space and consider M as a topological space with the topology stemming from the open balls (the ball topology).

Recall that a set A is closed if $M \setminus A$ is open.

Show that $A \subset M$ is closed if and only if

$$x_n \in A, \quad x_n \rightarrow x \quad \implies \quad x \in A.$$

Show that if (M, d) is a complete metric space and A is a closed subset of M , then (A, d) is a complete metric space.

Assume that A is closed and let $x_n \in A$ be a convergent sequence in M , i.e. $x_n \rightarrow x \in M$. We shall prove that $x \in A$.

INDIRECT PROOF. Assume that $x \notin A$, i.e. $x \in M \setminus A$, which is open.

There exists an $r > 0$, such that

$$B(x, r) \subseteq M \setminus A, \quad \text{i.e.} \quad B(x, r) \cap A = \emptyset.$$

Now, $x_n \rightarrow x$, so there exists an $n_r \in \mathbb{N}$, such that

$$d(x_n, x) < r \quad \text{for } n \geq n_r,$$

and we see that $x_n \in B(x, r) \cap A = \emptyset$, which is not possible. Hence our assumption is wrong, so we conclude that $x \in A$.

Conversely, assume for every convergent sequence $(x_n) \subseteq A$ the limit point lies in A . We shall prove that A is closed, or equivalently that $M \setminus A$ is open.

INDIRECT PROOF. Assume that $M \setminus A$ is *not* open. There exists an $x \in M \setminus A$, such that

$$\forall r \in \mathbb{R}_+ \exists y \in A : d(x, y) < r.$$

If we put $r = \frac{1}{n}$, $n \in \mathbb{N}$, with corresponding $y = x_n$, we define a sequence in A , which converges towards x , thus $x \in A$ according to the assumption. This is contradicting the assumption that $x \in M \setminus A$. Hence this assumption must be wrong, and $x \in A$ as requested.

Finally, assume that (M, d) is a *complete* metric space and that A is a *closed* subset of M . We shall prove that (A, d) is complete.

Let (x_n) be a Cauchy sequence on A . Then (x_n) is also a Cauchy sequence on the complete metric space M , thus (x_n) converges in M towards the limit $x \in M$. However, A is a closed subset, so it follows from the previous result that $x \in A$. We have proved that every Cauchy sequence (x_n) on A has a limit $x \in A$, which means that (A, d) is complete.

Example 1.11 Show that

$$d(x, y) = |\arctan x - \arctan y|$$

defines a metric on \mathbb{R} .

The definition includes an absolute value, hence $d(x, y) \geq 0$ for all $x, y \in \mathbb{R}$.

The function $\arctan t$ is strictly increasing on \mathbb{R} , hence $d(x, y) = 0$, if and only if $x = y$.

Clearly, $d(x, y) = d(y, x)$.

The triangle inequality follows from

$$d(x, y) = |\arctan x - \arctan y| \leq |\arctan x - \arctan z| + |\arctan z - \arctan y| = d(x, z) + d(z, y).$$

Example 1.12 In \mathbb{R}^k we define

$$d_1(x, y) = \sum_{i=1}^k |x_i - y_i|,$$

$$d_2(x, y) = \left(\sum_{i=1}^k |x_i - y_i|^2 \right)^{\frac{1}{2}},$$

$$d_\infty(x, y) = \max_{1 \leq i \leq k} |x_i - y_i|.$$

Show that d_1 , d_2 and d_∞ are metrics.

Show that

$$d_\infty(x, y) \leq d_1(x, y) \leq k d_\infty(x, y),$$

and find a similar inequality when d_1 is replaced by d_2 .

Show that if a sequence (x_n) converges to x in one of these metrics, then we have coordinate wise convergence:

$$x_{ni} \rightarrow x_i \quad \text{for all } i = 1, 2, \dots, k.$$

We first prove that

$$d_1(x, y) = \sum_{i=1}^k |x_i - y_i|$$

is a metric:

- 1) Clearly, $d_1(x, y) \geq 0$.
- 2) Clearly, $d_1(x, y) = 0$, if and only if $x = y$.
- 3) Clearly, $d_1(x, y) = d_1(y, x)$.

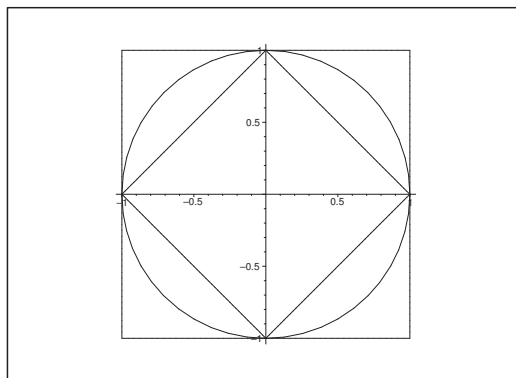


Figure 1: The three unit balls for d_1 (innermost), d_2 (the disc) and d_∞ (largest) in the case \mathbb{R}^2 .

4) The triangle inequality follows by a small computation

$$\begin{aligned} d_1(x, y) &= \sum_{i=1}^k |x_i - y_i| \leq \sum_{i=1}^k \{|x_i - z_i| + |z_i - y_i|\} \\ &= \sum_{i=1}^k |x_i - z_i| + \sum_{i=1}^k |z_i - y_i| = d_1(x, z) + d_1(z, y). \end{aligned}$$

We have proved that d_1 is a metric.

Then we prove that

$$d_2(x, y) = \left(\sum_{i=1}^k |x_i - y_i|^2 \right)^{\frac{1}{2}}$$

is a metric. Again, the first three conditions are trivial. The triangle inequality,

$$\sqrt{\sum_{i=1}^k |x_i - y_i|^2} \leq \sqrt{\sum_{i=1}^k |x_i - z_i|^2} + \sqrt{\sum_{i=1}^k |z_i - y_i|^2}$$

is, however, more difficult to prove. There are several proofs of the triangle inequality of d_2 . Here we shall not choose the most elegant one, but instead the intuitively most obvious one.

Put $a_i = x_i - z_i$ and $b_i = z_i - y_i$, $i = 1, \dots, k$. We shall prove that

$$\sqrt{\sum_{i=1}^k (a_i + b_i)^2} \leq \sqrt{\sum_{i=1}^k a_i^2} + \sqrt{\sum_{i=1}^k b_i^2}.$$

All terms are ≥ 0 , thus it is seen by squaring that we shall prove that

$$\sum_{i=1}^k a_i^2 + \sum_{i=1}^k b_i^2 + 2 \sum_{i=1}^k a_i b_i \leq \sum_{i=1}^k a_i^2 + \sum_{i=1}^k b_i^2 + 2 \sqrt{\sum_{i=1}^k \sum_{j=1}^k a_i^2 b_j^2},$$

which is reduced to the equivalent condition

$$\sum_{i=1}^k a_i b_i \leq \sqrt{\sum_{i=1}^k a_i^2} \cdot \sqrt{\sum_{j=1}^k b_j^2}.$$

The claim follows if we can prove the CAUCHY-SCHWARZ INEQUALITY

$$\left| \sum_{i=1}^k a_i b_i \right| \leq \sqrt{\sum_{i=1}^k a_i^2} \cdot \sqrt{\sum_{j=1}^k b_j^2}.$$

Another squaring shows that it suffices to prove that

$$\left(\sum_{i=1}^k a_i b_i \right)^2 \leq \sum_{i=1}^k a_i^2 \sum_{j=1}^k b_j^2,$$

i.e.

$$\sum_{i=1}^k a_i^2 b_i^2 + 2 \sum_{i=1}^{k-1} \sum_{j=i+1}^k a_i a_j b_i b_j \leq \sum_{i=1}^k a_i^2 b_i^2 + \sum_{i=1}^{k-1} \sum_{j=i+1}^k (a_i^2 b_j^2 + a_j^2 b_i^2),$$

which again is equivalent with

$$0 \leq \sum_{i=1}^{k-1} \sum_{j=i+1}^k (a_i^2 b_j^2 + a_j^2 b_i^2 - 2a_i a_j b_i b_j) = \sum_{i=1}^{k-1} \sum_{j=i+1}^k (a_i b_j - a_j b_i)^2.$$

The latter is clearly satisfied. Since we everywhere have computed “ \Leftarrow ”, the claim is proved.

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Finally,

$$d_\infty(x, y) = \max_{1 \leq i \leq k} |x_i - y_i|$$

is a metric, because the first three conditions again are trivial, and the triangle inequality follows from

$$|x_i - y_i| \leq |x_i - z_i| + |z_i - y_i| \quad \text{for every } i = 1, \dots, k,$$

thus

$$|x_i - y_i| \leq d_\infty(x, z) + d_\infty(z, y) \quad \text{for every } i = 1, \dots, k,$$

and by taking the maximum once more,

$$d_\infty(x, y) \leq d_\infty(x, z) + d_\infty(z, y).$$

We have now proved that d_1 , d_2 and d_∞ are all metrics.

We can find $j \in \{1, \dots, k\}$, such that

$$\begin{aligned} d_\infty(x, y) &= \max_{1 \leq i \leq k} |x_i - y_i| = |x_j - y_j| \leq \sum_{i=1}^k |x_i - y_i| = d_1(x, y) \\ &\leq \sum_{i=1}^k \max |x_i - y_i| = k \cdot d_\infty(x, y). \end{aligned}$$

Analogously (with the same “maximal” j),

$$\begin{aligned} d_\infty(x, y) &= \max_{1 \leq i \leq k} |x_i - y_i| = |x_j - y_j| = \sqrt{|x_j - y_j|^2} \\ &\leq \sqrt{\sum_{i=1}^k |x_i - y_i|^2} = d_2(x, y) \leq \sqrt{\sum_{i=1}^k \left\{ \max_{1 \leq i \leq k} |x_i - y_i| \right\}^2} \\ &= \sqrt{\sum_{i=1}^k \{d_\infty(x, y)\}^2} = \sqrt{k} \cdot d_\infty(x, y), \end{aligned}$$

and the wanted inequality becomes

$$d_\infty(x, y) \leq d_2(x, y) \leq \sqrt{k} \cdot d_\infty(x, y).$$

Remark 1.1 A simple squaring shows that $d_2(x, y) \leq d_1(x, y)$, which can also be seen on the figure (the simple proof is left to the reader). This means that

$$d_\infty(x, y) \leq d_2(x, y) \leq d_1(x, y) \leq k \cdot d_\infty(x, y). \quad \diamond$$

Using that $x_{ni} \rightarrow x_i$ for every $i = 1, 2, \dots, k$, if and only if $d_\infty(x_n, x) \rightarrow 0$, we conclude from the inequalities

$$d_\infty(x, y) \leq d_1(x, y) \leq k \cdot d_\infty(x, y),$$

$$d_\infty(x, y) \leq d_2(x, y) \leq \sqrt{k} \cdot d_\infty(x, y),$$

that this is fulfilled if and only if $d_1(x_n, 0) \rightarrow 0$, and if and only if $d_2(x, y) \rightarrow 0$.

Example 1.13 Let c denote the set of convergent complex sequences $x = (x_1, x_2, \dots)$. Show that c is a complete metric space when equipped with the metric

$$d_\infty(x, y) = \sup_i |x_i - y_i|.$$

HINT: Show that the space of bounded complex sequences ℓ^∞ is a complete space and show then that c is a closed subset, then apply Example 1.10.

Let $x^n = (x_1^n, x_2^n, \dots)$, where $\lim_{i \rightarrow \infty} x_i^n$ exists, be a Cauchy sequence from c , thus

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} \forall m, n \geq N : d(x^m, x^n) < \varepsilon.$$

This means that

$$\sup_i |x_i^m - x_i^n| < \varepsilon.$$

In particular, $(x_i^n)_n$ is a Cauchy sequence on \mathbb{R} for every i , hence convergent,

$$\lim_{n \rightarrow \infty} x_i^n = x_i.$$

The *Hint* is *not* used, because it is not hard to prove directly that $(x_i) \in c$. It suffices to prove that (x_i) is a Cauchy sequence, i.e.

$$(4) \quad \forall \varepsilon > 0 \exists I \in \mathbb{N} \forall i, j \geq I : |x_i - x_j| < \varepsilon.$$

It follows from

$$|x_i - x_j| \leq |x_i - x_i^n| + |x_i^n - x_j^n| + |x_j^n - x_j|,$$

and $(x_i^n)_n \rightarrow x_i$, and even

$$\sup_i |x_i - x_i^n| \rightarrow 0 \quad \text{for } n \rightarrow \infty,$$

that

$$a) \quad \forall \varepsilon > 0 \exists N \forall n \geq N \forall i : |x_i - x_i^n| < \frac{\varepsilon}{3},$$

$$b) \quad \forall \varepsilon > 0 \forall n \exists I(n) \forall i, j \geq I(n) : |x_i^n - x_j^n| < \frac{\varepsilon}{3}.$$

First choose N , such that a) is fulfilled.

Then choose $I = I(N)$, such that b) is fulfilled for $n = N$.

If $i, j \geq I = I(N)$, then

$$\begin{aligned} |x_i - x_j| &\leq |x_i - x_i^N| + |x_i^N - x_j^N| + |x_j^N - x_j| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon, \end{aligned}$$

which is (4), and we have proved that (x_i) is a Cauchy sequence on \mathbb{R} , hence convergent. In particular, (x_i) is bounded, so $(x_i) \in c$, and c is complete.

Example 1.14 In the set of bounded complex sequences ℓ^∞ equipped with the metric from EXERCISE 12 we consider the sets c_0 consisting of the sequences converging to 0 and c_{00} consisting of the sequences with only a finite number of elements different from 0. Investigate if c_0 and/or c_{00} are closed subsets of ℓ^∞ .

The sequence $\left(\frac{1}{n}\right)$ belongs to ℓ^∞ , though it does not belong to c_{00} . Choose

$$x^n = \left(1, \frac{1}{2}, \dots, \frac{1}{n}, 0, 0, \dots\right).$$

Then $x^n \in c_{00}$ and $x^n \rightarrow x = \left(\frac{1}{n}\right) \notin c_{00}$, hence c_{00} is not closed.

Let $x^n = (x_1^n, x_2^n, \dots) \in c_0$ be convergent in ℓ^∞ , i.e. $\lim_{i \rightarrow \infty} x_i^n = 0$ for every n . There exists an $x \in \ell^\infty$, such that

$$\forall \varepsilon > 0 \exists n_0 \forall n \geq n_0 : \|x - x^n\|_\infty = \sup_i |x_i - x_i^n| < \varepsilon.$$

We shall prove that $\lim_{i \rightarrow \infty} x_i = 0$. Now,

$$|x_i| \leq |x_i - x_i^n| + |x_i^n| \leq \|x - x^n\|_\infty + |x_i^n|.$$

First choose n , such that $\|x - x^n\|_\infty < \frac{\varepsilon}{2}$.

Then choose I , such that $|x_i^n| < \frac{\varepsilon}{2}$ for every $i \geq I$. Summing up we get for all $i \geq I$ that

$$|x_i| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

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1.3 Contractions

Example 1.15 Consider the metric space (M, d) , where $M = [1, \infty[$, and d the usual distance. Let the mapping $T : M \rightarrow M$ be given by

$$Tx = \frac{x}{2} + \frac{1}{x}.$$

Show that T is a contraction and find the minimal contraction constant α . Find also the fixed point.

First compute

$$|Tx - Ty| = \left| \frac{x}{2} + \frac{1}{x} - \frac{y}{2} - \frac{1}{y} \right| = \left| \frac{x-y}{2} + \frac{1}{x} - \frac{1}{y} \right| = \left| \frac{x-y}{2} + \frac{y-x}{xy} \right| = |x-y| \cdot \left| \frac{1}{2} - \frac{1}{xy} \right|.$$

Now, $x, y \geq 1$, so $0 < \frac{1}{xy} \leq 1$, and the function

$$(x, y) \mapsto \frac{1}{2} - \frac{1}{xy}$$

has the range $\left[-\frac{1}{2}, \frac{1}{2}\right]$. We conclude that $\alpha = \frac{1}{2}$, so $\frac{1}{2}$ is the smallest α , for which

$$\left| \frac{1}{2} - \frac{1}{xy} \right| \leq \alpha.$$

The fixpoint satisfies the equation $Tx = x$, thus

$$x = \frac{x}{2} + \frac{1}{x}, \quad \text{hence } \frac{x}{2} = \frac{1}{x}, \quad \text{i.e. } x^2 = 2.$$

Since $x \geq 1$, the fixpoint must be $x = \sqrt{2}$, which also is easily seen by insertion.

Since $\alpha = \frac{1}{2} < 1$, it follows from the above that it is the only fixpoint.

Example 1.16 A mapping T from a metric space (M, d) into itself is called a weak contraction if

$$d(Tx, Ty) < d(x, y),$$

for all $x, y \in M$, $x \neq y$.

Show that T has at most one fixed point.

Show that T does not necessarily have a fixed point.

HINT: One should take $Tx = x + \frac{1}{x}$ for $x \geq 1$.

Let T be a weak contraction, and assume that both x and y are fixpoints, i.e. $Tx = x$ and $Ty = y$.

If $x \neq y$, then

$$d(x, y) = d(Tx, Ty) < d(x, y),$$

which is not possible. Hence $y = x$, and there is at most one fixpoint.

Define $Tx = x + \frac{1}{x}$ on $[1, +\infty[$. If $x, y \in [1, +\infty[$, then

$$|Tx - Ty| = \left| x + \frac{1}{x} - y - \frac{1}{y} \right| = \left| x - y + \frac{y - x}{xy} \right| = |x - y| \cdot \left| 1 - \frac{1}{xy} \right|.$$

It follows from $0 < \frac{1}{xy} \leq 1$ for $x, y \geq 1$, that

$$|Tx - Ty| < |x - y| \quad \text{for } x \neq y,$$

and T is a weak contraction on $[1, +\infty[$.

The weak contraction $Tx = x + \frac{1}{x}$ does not have a fixpoint, because $Tx = x$ would imply that $\frac{1}{x} = 0$, which is not possible.

Example 1.17 *It is very common in mathematical analysis to consider iterations of the form*

$$x_n = g(x_{n-1}),$$

where g is a C^1 -function. Show that the sequence (x_n) is convergent for any choice of x_0 if there is an α , $0 < \alpha < 1$, such that

$$|g'(x)| \leq \alpha$$

for all $x \in \mathbb{R}$.

It follows from the Mean Value Theorem that one to any x and y can find $t = t(x, y)$ between x and y , such that

$$g(x) - g(y) = g'(t) \cdot (x - y),$$

thus

$$|g(x) - g(y)| = |g'(t)| \cdot |x - y| \leq \alpha |x - y|.$$

This proves that g is a contraction, and the claim follows from Banach's Fixpoint Theorem.

Example 1.18 *To approximate the solution to an equation $f(x) = 0$, we bring the equation on the form $x = g(x)$ and choose an x_0 and use the iteration $x_n = g(x_{n-1})$. Assume that g is a C^1 -function on the interval $[x_0 - \delta; x_0 + \delta]$, and that $|g'(x)| \leq \alpha < 1$ for $x \in [x_0 - \delta; x_0 + \delta]$, and moreover*

$$|g(x_0) - x_0| \leq (1 - \alpha)\delta.$$

Show that there is one and only one solution $x \in [x_0 - \delta; x_0 + \delta]$ to the equation, and that $x_n \rightarrow x$.

Noticing that $|g'(x)| \leq \alpha < 1$ on the interval $[x_0 - \delta; x_0 + \delta]$, the claim follows from Banach's Fixpoint Theorem, if we only can prove that the iterative sequence (x_n) lies entirely in the interval $[x_0 - \delta, x_0 + \delta]$. We prove this by induction.

It is obvious that $x_0 \in [x_0 - \delta, x_0, \delta]$.

Assume that $x_n \in [x_0 - \delta, x_0 + \delta]$. Then we get for the following element $x_{n+1} = g(x_n)$,

$$\begin{aligned} |x_{n+1} - x_0| &= |g(x_n) - x_0| \\ &\leq |g(x_n) - g(x_0)| + |g(x_0) - x_0| \\ &\leq \alpha |x_n - x_0| + (1 - \alpha)\delta \\ &\leq \alpha \delta + (1 - \alpha)\delta = \delta, \end{aligned}$$

proving that $x_{n+1} \in [x_0 - \delta, x_0 + \delta]$, and the claim follows.

Example 1.19 Solve by iteration the equation $f(x) = 0$ for $f \in C^1([a, b])$, $f(x) < 0 < f(b)$ and f' bounded and strictly positive in $[a, b]$.

HINT: Take $g(x) = x - \lambda f(x)$ for a smart choice of λ .

Putting

$$g(x) = x - \lambda f(x), \quad \lambda \neq 0,$$

it follows that $f(x) = 0$, if and only if $g(x) = x$. Now,

$$g'(x) = 1 - \lambda f'(x) \quad \text{and} \quad 0 < k_1 \leq f'(x) \leq k_2,$$

so

$$1 - \lambda k_2 \leq g'(x) \leq 1 - \lambda k_1.$$

If we choose $\lambda = \frac{1}{k_2}$, then

$$0 \leq g'(x) \leq 1 - \frac{k_1}{k_2} = \alpha < 1,$$

and the mapping $g : [a, b] \rightarrow [a, b]$ is increasing and a contraction, so it has by Banach's Fixpoint Theorem precisely one fixpoint in $[a, b]$.

Example 1.20 Show that it is possible to solve the equation $f(x)x^3 + x - 1 = 0$ by the iteration

$$x_n = g(x_{n-1}) = (1 + x_{n-1}^2)^{-1}.$$

Find x_1, x_2, x_3 for $x_0 = 1$, and find an estimate for $d(x, x_n)$.

Let $g(x) = \frac{1}{1 + x^2}$. Then $g(x) = x$ is equivalent with $x = \frac{1}{1 + x^2}$, thus $x(1 + x^2) = 1$, which we write as

$$f(x) = x^3 + x - 1 = 0,$$

i.e. exactly the equation we want to solve.

It follows from

$$g'(x) = -\frac{2x}{(1 + x^2)^2},$$

and

$$g''(x) = -\frac{2}{(1+x^2)^2} - 2x \cdot \frac{(-2) \cdot 2x}{(1+x^2)^3} = \frac{2}{(1+x^2)^3} \{-1 - x^2 + 4x^2\} = \frac{6\left(x^2 - \frac{1}{3}\right)}{(1+x^2)^3},$$

that $g''(x) = 0$ for $x = \pm \frac{1}{\sqrt{3}}$. Since $g'(x) \rightarrow 0$ for $x \rightarrow \pm\infty$, these points correspond to maximum and minimum for $g'(x)$, thus

$$|g'(x)| \leq \frac{2 \cdot \frac{1}{\sqrt{3}}}{\left(1 + \frac{1}{3}\right)^2} = \frac{\frac{2}{\sqrt{3}}}{\frac{16}{9}} = \frac{3\sqrt{3}}{8} = \alpha \leq 0.65,$$

and we have proved that g is a contraction, so the equation

$$f(x) = x^3 + x - 1 = 0$$

can be solved by the given iteration.

Let $x_0 = 1$. Then

$$x_1 = g(x_0) = \frac{1}{1+1} = \frac{1}{2},$$

$$x_2 = g\left(\frac{1}{2}\right) = \frac{1}{1 + \frac{1}{4}} = \frac{4}{5},$$

$$x_3 = g\left(\frac{4}{5}\right) = \frac{1}{1 + \frac{16}{25}} = \frac{25}{41}.$$

Finally,

$$|x - x_n| \leq \frac{\alpha^n}{1 - \alpha} \cdot |x_1 - x_0|,$$

so

$$|x - x_n| \leq \frac{\left(\frac{3\sqrt{3}}{8}\right)^n}{1 - \frac{3\sqrt{3}}{8}} \cdot \left(1 - \frac{1}{2}\right) = \frac{4}{8 - 3\sqrt{3}} \cdot \left(\frac{3\sqrt{3}}{8}\right)^n = \frac{4}{8 - 3\sqrt{3}} \cdot \left(\frac{27}{64}\right)^{\frac{n}{2}} < \frac{3}{2} \cdot \left(\frac{27}{64}\right)^{\frac{n}{2}}.$$

When we apply the iteration above on a pocket calculator, we get

$$x = 0.682\,327\,804.$$

Remark 1.2 The iteration above can therefore be applied, though it is far from the fastest one. If the preset case we get by *Newton's iteration formula*

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = \frac{2}{3}x_n + \frac{1}{3} \cdot \frac{3 - 2x_n}{3x_n^2 + 1},$$

from which already

$$x_4 = 0.682\,327\,804. \quad \diamond$$

Example 1.21 A mapping $T : \mathbb{R} \rightarrow \mathbb{R}$ satisfies a Lipschitz condition with constant k , if

$$|Tx - Ty| \leq k|x - y|, \quad \text{for all } x, y \in \mathbb{R}.$$

- 1) Is T a contraction?
- 2) If T is a C^1 -function with bounded derivative, show that T satisfies a Lipschitz condition.
- 3) If T satisfies a Lipschitz condition, is T then a C^1 -function with bounded derivative?
- 4) Assume that $|Tx - Ty| \leq k|x - y|^\alpha$ for some $\alpha > 1$. Show that T is a constant.

- 1) If $k \geq 1$, then T is not necessarily a contraction.
If instead $0 \leq k < 1$, then T is always a contraction.

- 2) It follows from the Mean Value Theorem that

$$|T(x) - T(y)| = |T'(t)| \cdot |x - y|,$$

where $t = t(x, y)$ lies somewhere between x and y .

Since $|T'(t)| \leq k$, it is obvious that T fulfils a Lipschitz condition.

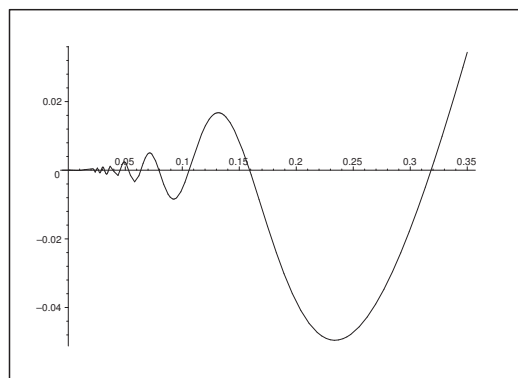


Figure 2: The graph of $f(x) = x^2 \cdot \sin \frac{1}{x}$ for $0 < x < 0.35$.

3) The answer is “no”. Choose the function

$$f(x) = \begin{cases} x^2 \cdot \sin \frac{1}{x} & \text{for } x > 0, \\ 0 & \text{for } x \leq 0. \end{cases}$$

Then f is differentiable with the derivative

$$f'(x) = \begin{cases} 2x \cdot \sin \frac{1}{x} - \cos \frac{1}{x} & \text{for } x > 0, \\ \lim_{x \rightarrow 0^+} \frac{f(x) - f(0)}{x - 0} = \lim_{x \rightarrow 0^+} x \cdot \sin \frac{1}{x} = 0 & \text{for } x = 0, \\ 0 & \text{for } x < 0. \end{cases}$$

Choose $x_0 > 0$, such that $f'(x_0) = 0$, and put

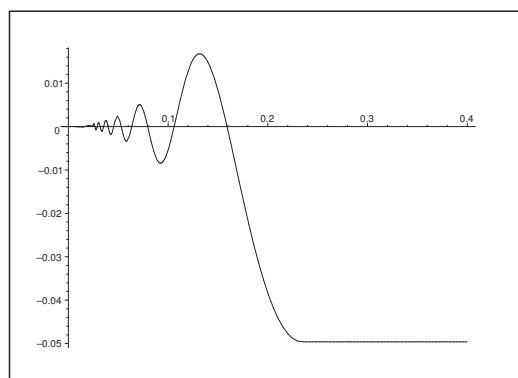


Figure 3: An example of a function $T(x)$.

$$T(x) = \begin{cases} f(x_0) & \text{for } x \geq x_0, \\ x^2 \cdot \sin \frac{1}{x} & \text{for } 0 < x < x_0, \\ 0 & \text{for } x \leq 0. \end{cases}$$

Then $|T'(x)| \leq 2x_0 + 1$, and $T'(x)$ is defined everywhere, though not continuous for $x = 0$, where $T'(x) = f'(x) = 2x \cdot \sin \frac{1}{x} - \cos \frac{1}{x}$ or $0 < x < x_0$ does not have a limit value for $x \rightarrow 0+$. Thus we have constructed a mapping $T \notin C^1$, which satisfies a Lipschitz condition. (It is of course possible to construct far more complicated examples).

4) Assume that there exists an $\alpha > 1$, such that

$$|Tx - Ty| \leq k|x - y|^\alpha.$$

Then

$$0 \leq \left| \lim_{y \rightarrow x} \frac{Tx - Ty}{x - y} \right| \leq \lim_{y \rightarrow x} k \cdot \frac{|x - y|^\alpha}{|x - y|} = k \cdot \lim_{y \rightarrow x} |y - x|^{\alpha-1} = 0.$$

This proves that T is differentiable everywhere of the derivative 0. Then T is a constant.

Example 1.22 Let T be a mapping from a complete metric space (M, d) into itself, and assume that there is a natural number m such that T^m is a contraction. Show that T has one and only one fixed point.

If T^m is a contraction, then T^m has a fixpoint x , thus $T^m x = x$. When we apply T on this equation, we get

$$T^{m+1}x = T^m(Tx) = Tx,$$

hence Tx is also a fixpoint of T^m .

Since T^m is a contraction, the fixpoint is unique, so $Tx = x$, and we have proved that x is a fixpoint for T .

Conversely, if x is a fixpoint for T , then x is also a fixpoint for T^m , because $Tx = x$ implies that

$$T^m x = T^{m-1}(Tx) = T^{m-1}x = \dots = Tx = x.$$

We have assumed that T^m is a contraction, hence the fixpoint for T^m is unique. This is true for every fixpoint x for T , hence it must be unique.

Example 1.23 We consider the metric space \mathbb{R}^k with the metric

$$d_1(x, y) = \sum_{i=1}^k |x_i - y_i|$$

and a mapping $T : \mathbb{R}^k \rightarrow \mathbb{R}^k$ given by $Tx = Cx + b$, where $C = (c_{ij})$ is a $k \times k$ matrix and $b \in \mathbb{R}^k$. Show that T is a contraction, if

$$\sum_{i=1}^k |c_{ij}| < 1 \quad \text{for all } j = 1, 2, \dots, k.$$

If we instead use the metric

$$d_2(x, y) = \sqrt{\sum_{i=1}^k |x_i - y_i|^2},$$

show that T is a contraction if

$$\sum_{i=1}^k \sum_{j=1}^k |c_{ij}|^2 < 1.$$

First note that the i -th coordinate of Tx is

$$(Tx)_i = \sum_{j=1}^k c_{ij}x_j + b_i, \quad i = 1, \dots, k.$$

Put $y = Tx$ and $w = Tz$ and

$$\alpha = \max_{1 \leq j \leq k} \sum_{i=1}^k |c_{ij}| < 1.$$

Then we get the estimates

$$\begin{aligned} d_1(Tx, Tz) &= \sum_{i=1}^k |y_i - w_i| = \sum_{i=1}^k \left| \sum_{j=1}^k c_{ij}(x_j - z_j) \right| \\ &\leq \sum_{i=1}^k \sum_{j=1}^k |c_{ij}| \cdot |x_j - z_j| \leq \alpha \sum_{j=1}^k |x_j - z_j| = \alpha \cdot d_1(x, z), \end{aligned}$$

and the condition $\alpha = \max_{1 \leq j \leq k} |c_{ij}| < 1$ assures that T is a contraction in (\mathbb{R}^k, d_1) .

If instead we consider the metric

$$d_2(x, y) = \sqrt{\sum_{i=1}^k |x_i - y_i|^2},$$

and assume that

$$\alpha^2 = \sum_{i=1}^k \sum_{j=1}^k |c_{ij}|^2 < 1,$$

then we get the following estimate

$$\begin{aligned} \{d_2(x, y)\}^2 &= \sum_{i=1}^k |y_i - w_i|^2 = \sum_{i=1}^k \left| \sum_{j=1}^k c_{ij}(x_j - z_j) \right|^2 \\ &= \sum_{i=1}^k \left| \sum_{j=1}^k c_{ij}(x_j - z_j) \cdot \sum_{\ell=1}^k c_{i\ell}(x_\ell - z_\ell) \right|^2 \\ &\leq \sum_{i=1}^k \sum_{j=1}^k \sum_{\ell=1}^k |c_{ij}| \cdot |x_j - z_j| \cdot |c_{i\ell}| \cdot |x_\ell - z_\ell|. \end{aligned}$$

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Then apply

$$|ab| \leq \frac{1}{2} (a^2 + b^2),$$

which follows from the inequality $(|a| - |b|)^2 = a^2 + b^2 - 2|ab| \geq 0$.

If we put

$$a = |c_{i\ell}| \cdot |x_j - z_j| \quad \text{and} \quad b = |c_{ij}| \cdot |x_\ell - z_\ell|,$$

we get

$$\begin{aligned} \{d_2(y, w)\}^2 &= \sum_{i=1}^k \sum_{j=1}^k \sum_{\ell=1}^k \frac{1}{2} \{ |c_{i\ell}|^2 |x_j - z_j|^2 + |c_{ij}|^2 |x_\ell - z_\ell|^2 \} \\ &= \frac{1}{2} \sum_{i=1}^k \sum_{\ell=1}^k |c_{i\ell}|^2 \cdot \sum_{j=1}^k |x_j - z_j|^2 + \frac{1}{2} \sum_{i=1}^k \sum_{j=1}^k k |c_{ij}|^2 \cdot \sum_{\ell=1}^k |x_\ell - z_\ell|^2 \\ &\leq \frac{1}{2} \alpha^2 \{d_2(x, z)\}^2 + \frac{1}{2} \alpha^2 \{d_2(x, z)\}^2 = \alpha^2 \{d_2(x, z)\}^2. \end{aligned}$$

Since $\alpha^2 < 1$, and hence also $0 \leq \alpha < 1$, and

$$d_2(y, w) = d_2(Tx, Tz) \leq \alpha \cdot d_2(x, z),$$

we conclude that T is a contraction in (\mathbb{R}^k, d_2) .

Example 1.24 *In connection with Banach's Fixpoint Theorem, the inequality*

$$d(x, x_n) \leq \frac{\alpha}{1 - \alpha} d(x_{n-1}, x_n)$$

is often mentioned. Prove this inequality.

Given that $\alpha \in]0, 1[$, at $Tx_n = x_{n+1}$, and $x_n \rightarrow x$.

Choose to any $\varepsilon \in \mathbb{R}_+$ an N , such that we for all $p \geq N$ have $d(x, x_p) < \varepsilon$. If $p \geq N$ and $p \geq n + 1$, then

$$\begin{aligned} d(x, x_n) &\leq d(x, x_p) + d(x_p, x_n) < \varepsilon + d(x_p, x_n) \\ &\leq \varepsilon + d(x_p, x_{p-1}) + d(x_{p-1}, x_{p-2}) + \cdots + d(x_{n+1}, x_n) \\ &= \varepsilon + d(Tx_{p-1}, Tx_{p-2}) + d(Tx_{p-2}, Tx_{p-3}) + \cdots + d(Tx_n, Tx_{n-1}) \\ &\leq \varepsilon + \alpha \cdot \frac{1 - \alpha^{p-n}}{1 - \alpha} \cdot d(x_{n-1}, x_n) \\ &\leq \varepsilon + \frac{\alpha}{1 - \alpha} \cdot d(x_{n-1}, x_n). \end{aligned}$$

This is true for every $\varepsilon > 0$, thus

$$d(x, x_n) \leq \frac{\alpha}{1 - \alpha} \cdot d(x_{n-1}, x_n).$$

Example 1.25 Consider the matrix equation $Ax + b = 0$, where $A = (a_{ij})_{i,j=1}^k$ (and the a_{ij} real). Put $A = C - I$ and rewrite the equation as $x = Cx + b$.

If

$$(5) \sum_{j=1}^k |c_{ij}| < 1 \quad \text{for } i = 1, 2, \dots, k,$$

then there is a unique solution x , which can be found by iteration.

Prove that the condition (5) can be formulated as the following condition of the a_{ij} ,

$$a_{ii} < 0, \quad |a_{ii}| > \sum_{j=1, j \neq i}^k |a_{ij}|, \quad |a_{ii}| < 2 - \sum_{j=1, j \neq i}^k |a_{ij}|,$$

for $i = 1, 2, \dots, k$.

We have $a_{ij} = c_{ij} - \delta_{ij}$, thus $c_{ij} = \delta_{ij} + a_{ij}$. In particular, $c_{ii} = 1 + a_{ii}$. Since

$$\sum_{j=1}^k |c_{ij}| < 1,$$

we have $|c_{ii}| < 1$, thus $a_{ii} \in]-2, 0[$.

Furthermore, $c_{ij} = |a_{ij}|$ for $i \neq j$, so

$$\sum_{j=1}^k |c_{ij}| = \sum_{j=1, j \neq i}^k |a_{ij}| + |1 + a_{ii}| < 1.$$

It follows that

$$\sum_{j=1}^k |a_{ij}| < 1 - |1 + a_{ii}| = 1 - |1 - |a_{ii}|| \leq 1.$$

If

$$|a_{ii}| \leq 1 \quad \left(< 2 - \sum_{j=1, j \neq i}^k |a_{ij}| \right),$$

then

$$\sum_{j=1, j \neq i}^k |a_{ij}| < 1 - 1 + |a_{ii}| = |a_{ii}|.$$

If

$$|a_{ii}| > 1 \quad \left(> \sum_{j=1, j \neq i}^k |a_{ij}| \right),$$

then

$$\sum_{j=1, j \neq i}^k |a_{ij}| < 1 - |a_{ii}| + 1 = 2 - |a_{ii}|,$$

hence by a rearrangement,

$$|a_{ii}| < 2 - \sum_{j=1, j \neq i}^k |a_{ij}|,$$

and we derive in both cases that

$$\sum_{j=1, j \neq i}^k |a_{ij}| < |a_{ii}| < 2 - \sum_{j=1, j \neq i}^k |a_{ij}|.$$

Conversely, assume that $a_{ii} < 0$ and that

$$\sum_{j=1, j \neq i}^k |a_{ij}| < |a_{ii}| < 2 - \sum_{j=1, j \neq i}^k |a_{ij}|.$$

Then

$$\sum_{j=1, j \neq i}^k |a_{ij}| < 1.$$

If $|a_{ii}| \leq 1$, then

$$|a_{ii}| = 1 - 1 + |a_{ii}| = 1 - |1 - |a_{ii}|| = 1 - |1 + a_{ii}| = 1 - |c_{ii}|,$$

thus

$$\sum_{j=1, j \neq i}^k |a_{ij}| = \sum_{j=1, j \neq i}^k |c_{ij}| < 1 - |c_{ii}|,$$

and hence

$$\sum_{j=1}^k |c_{ij}| < 1.$$

If $|a_{ii}| > 1$, then

$$|a_{ii}| = 1 - 1 + |a_{ii}| = 1 + ||a_{ii}| - 1| = 1 + |a_{ii} + 1| = 1 + |c_{ii}|,$$

hence by insertion

$$1 + |c_{ii}| < 2 - \sum_{j=1, j \neq i}^k |a_{ijn}| = 2 - \sum_{j=1, j \neq i}^k |c_{ij}|,$$

follows by a rearrangement

$$\sum_{j=1}^k |c_{ij}| < 1.$$

1.4 Simple integral equations

Example 1.26 Consider the Volterra integral equation:

$$x(t) - \mu \int_a^t k(t, s)x(s) ds = v(t), \quad t \in [a, b],$$

where $v \in C([a, b])$, $k \in C([a, b]^2)$ and $\mu \in \mathbb{C}$.

Show that the equation has a unique solution $x \in C([a, b])$ for any $\mu \in \mathbb{C}$.

HINT: Write the equation $x = Tx$ where

$$Tx = v(t) + \mu \int_a^t k(t, s)x(s) ds.$$

Take $x_0 \in C([a, b])$ and define the iteration by $x_{n+1} = Tx_n$, then show by induction that

$$|T^m x(t) - T^m y(t)| \leq |\mu|^m c^m \frac{(t-a)^m}{m!} d_\infty(x, y),$$

where $c = \max |k|$. Then show (by looking at $d_\infty(T^m x, T^m y)$) that T^m is a contraction for some m and argue that T then must have a unique fixed point in the metric space $(C([a, b]), d_\infty)$.

Using the given definition of T we see that the equation is equivalent with $Tx = x$. Then

$$\begin{aligned} |Tx(t) - Ty(t)| &= |\mu| \cdot \left| \int_a^t k(t, s)x(s) ds - \int_a^t k(t, s)y(s) ds \right| = |\mu| \cdot \left| \int_a^t k(t, s) \cdot \{x(s) - y(s)\} ds \right| \\ &\leq |\mu| \cdot c \cdot d_\infty(x - y) \cdot \left| \int_a^t 1 ds \right| = |\mu|^1 \cdot c^1 \cdot \frac{(t-a)^1}{1!} d_\infty(x, y), \end{aligned}$$

which shows that the inequality above holds for $m = 1$.

Assume that for some $m \in \mathbb{N}$,

$$(6) \quad |T^m x(t) - T^m y(t)| \leq |\mu|^m c^m \cdot \frac{(t-a)^m}{m!} d_\infty(x, y).$$

Then

$$\begin{aligned} |T^{m+1} x(t) - T^{m+1} y(t)| &= |\mu| \cdot \left| \int_a^t k(t, s)\{T^m x(s) - T^m y(s)\} ds \right| \\ &\leq |\mu| \cdot c \int_a^t |T^m x(s) - T^m y(s)| ds \\ &\leq |\mu| \cdot c \cdot |\mu|^m \cdot c^m \cdot d_\infty(x, y) \cdot \int_a^t \frac{(s-a)^m}{m!} ds \\ &= |\mu|^{m+1} c^{m+1} \cdot d_\infty(x, y) \cdot \left[\frac{(s-a)^{m+1}}{(m+1)!} \right]_a^t \\ &= |\mu|^{m+1} c^{m+1} \cdot \frac{(t-a)^{m+1}}{(m+1)!} \cdot d_\infty(x, y), \end{aligned}$$

and (6) follows by induction for all $m \in \mathbb{N}$.

We infer from (6) that

$$d_\infty(T^m x, T^m y) \leq |\mu|^m c^m \cdot \frac{(b-a)^m}{m!} \cdot d_\infty(x, y).$$

Now

$$\sum_{m=0}^{\infty} |\mu|^m c^m \cdot \frac{(b-a)^m}{m!} = \exp(|\mu| \cdot c \cdot (b-a))$$

is convergent, thus

$$|\mu|^m c^m \cdot \frac{(b-a)^m}{m!} \rightarrow 0 \quad \text{for } m \rightarrow \infty.$$

There exists in particular an $M \in \mathbb{N}$, such that

$$\alpha = |\mu|^m c^m \cdot \frac{(b-a)^m}{m!} < 1 \quad \text{for all } m \geq M.$$

Thus, if $m \geq M$, then T^m is a contraction, and T^m has a fixpoint x . An application of EXAMPLE 1.22 shows that x is also a fixpoint for T , and x is the unique fixpoint of T .

Let $x_0 \in C^0([a, b])$. Define by iteration $x_{m+1} = Tx_m$. Then $x_m = T^m x_0$. The sequence $(x_{m \cdot n})$ converges towards x . The same does the sequence (x_{mn+j}) , where $j = 0, 1, \dots, m-1$, because

$$x_{mn+j} = T^{mn} (T^j x_0) = T^{mn+j} x_0.$$

Summing up we conclude that (x_n) itself converges towards x , and the claim is proved.

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Example 1.27 Solve by iteration the equation

$$f(t) = u(t) = \frac{1}{2} \int_0^1 e^{t-s} f(s) ds, \quad t \in [0, 1],$$

(where u is a given continuous function), by choosing f_0 as u .
Find in particular the solutions in the cases

$$u(t) = 1, \quad u(t) = t.$$

Then solve the equation directly (without using iteration), assuming that $u \in C^1([0, 1])$.

If we put $f_0(t) = u(t)$, then

$$f_1(t) = u(t) + \frac{1}{2} \int_0^1 e^{t-s} u(s) ds = u(t) + \frac{1}{2} \left\{ \int_0^1 e^{-s} u(s) ds \right\} \cdot e^t.$$

Putting $a = \int_0^1 e^{-s} u(s) ds$, we get

$$f(t) = u(t) + \frac{a}{2} e^t.$$

It follows that

$$\begin{aligned} f_2(t) &= u(t) + \frac{1}{2} \int_0^1 e^{t-s} f_1(s) ds = u(t) + \frac{1}{2} e^t \left\{ \int_0^1 e^{-s} u(s) ds + \frac{a}{2} \int_0^1 e^{-s} e^s ds \right\} \\ &= u(t) + e^t \left\{ \frac{a}{2} + \frac{a}{4} \right\} = u(t) + \frac{3}{4} a \cdot e^t. \end{aligned}$$

We conclude from the structure

$$f(t) = u(t) + e^t \left\{ \frac{1}{2} \int_0^1 e^{-s} f(s) ds \right\},$$

that a solution must have the form $f(t) = u(t) + c \cdot e^t$. We therefore guess that the n -th iteration may be written

$$f_n(t) = u(t) + a \cdot k_n e^t.$$

We get by insertion

$$\begin{aligned} f_{n+1}(t) &= u(t) + \frac{1}{2} \int_0^1 e^{t-s} f_n(s) ds \\ &= u(t) + \frac{1}{2} e^t \left\{ \int_0^1 e^{-s} u(s) ds + a \cdot k_n \int_0^1 e^{-s} e^s ds \right\} \\ &= u(t) + \frac{1}{2} a e^t \left\{ \frac{1}{2} e^t \cdot a \cdot k_n \right\} = u(t) + a \left\{ \frac{1+k_n}{2} \right\} e^t, \end{aligned}$$

and conclude that

$$k_{n+1} = \frac{1}{2} (1 + k_n).$$

If $k_n \in [0, 1[$, then it follows that $k_n < k_{n+1} < 1$, thus (k_n) is increasing and bounded. (Notice that $k_1 = \frac{1}{2}$), thus it is convergent of the limit value k . We conclude from the equation of recursions that $k = \frac{1}{2}(1 + k)$, thus $k = 1$. Hence the solution is given by

$$f(t) = u(t) + e^t \int_0^1 e^{-s} u(s) ds.$$

CHECK. We get by insertion,

$$u(t) + \frac{1}{2} \int_0^1 e^{t-s} f(s) ds = u(t) + \frac{1}{2} e^t \int_0^1 e^{-s} u(s) ds + \frac{1}{2} e^t \int_0^1 e^{-s} u(s) ds = f(t),$$

proving that we have found a solution. \diamond

If $u(t) = 1$, then

$$f(t) = 1 + e^t \int_0^1 e^{-s} ds = 1 + e^t [-e^{-s}]_0^1 = 1 + \left(1 - \frac{1}{e}\right) e^t.$$

If $u(t) = t$, then

$$f(t) = t + e^t \int_0^1 s e^{-s} ds = t + e^t [-s e^{-s} - e^{-s}]_0^1 = t + \left(1 - \frac{2}{e}\right) e^t.$$

As mentioned above the solution must have the form $u(t) + c \cdot e^t$. Then by insertion,

$$\begin{aligned} u(t) + \frac{1}{2} \int_0^1 e^{t-s} f(s) ds &= u(t) + \frac{1}{2} \int_0^1 e^{t-s} \{u(s) + c \cdot e^s\} ds \\ &= u(t) + \frac{1}{2} \left\{ \int_0^1 e^{-s} u(s) ds + c \right\} e^t = u(t) + c \cdot e^t = f(t), \end{aligned}$$

and we conclude that $c = \int_0^1 e^{-s} u(s) ds$.

If $u \in C^1([0, 1])$, then

$$f(t) = u(t) + \left\{ \frac{1}{2} \int_0^1 e^{-s} f(s) ds \right\} \cdot e^t \in C^1,$$

so we can ALTERNATIVELY solve the equation by differentiation with respect to t . It follows from

$$\frac{1}{2} \int_0^1 e^{t-s} f(s) ds = f(t) - u(t),$$

that

$$f'(t) = u'(t) + \frac{1}{2} \int_0^1 e^{t-s} f(s) ds = f(t) + u'(t) - u(t),$$

hence by a multiplication by e^{-t} follows by a rearrangement,

$$f'(t) e^{-t} - f(t) e^{-t} = \frac{d}{dt} \{e^{-t} f(t)\} = u'(t) e^{-t} - u(t) e^{-t} = \frac{d}{dt} \{e^{-t} u(t)\},$$

and we get by an integration

$$e^{-t}f(t) = e^{-t}u(t) + c,$$

hence

$$f(t) = u(t) + c \cdot e^t.$$

The constant c is determined as above. The latter variant is of course not the shortest one.

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Example 1.28 Let $C^0([a, b])$ be equipped with the metric

$$d(x, y) = \max_{t \in [a, b]} |x(t) - y(t)|.$$

We define an operator (a mapping) S by

$$Sx(t) = \int_a^b k(t, s)x(s) ds,$$

where k is a continuous function on $[a, b] \times [a, b]$. Let (x_n) be inductively given by

$$(7) \quad x_{n+1} = u + \mu Sx_n,$$

and put $z_n = x_n - x_{n-1}$. Prove that (7) equivalently can be written

$$(8) \quad z_{n+1} = \mu Sz_n.$$

Put $x_0 = u$, and prove that (7) implies the Neumann series

$$x = \lim_{n \rightarrow \infty} x_n = u + \mu Su + \mu^2 S^2 u + \cdots .$$

We note that

$$x_{n+1}(t) = u(t) + \mu \int_a^b k(t, s)x_n(s) ds = u(t) + \mu Sx_n(t).$$

Putting $z_n = x_n - x_{n-1}$, we get

$$\begin{aligned} z_{n+1} &= x_{n+1} - x_n = u + \mu Sx_n - u - \mu Sx_{n-1} \\ &= \mu S(x_n - x_{n-1}) = \mu Sz_n. \end{aligned}$$

If $|\mu| < \frac{1}{(b-a)c}$, then $x_n \rightarrow x$. It follows from

$$x_n = x_n - x_{n-1} + x_{n-1} - x_{n-2} + x_{n-2} + \cdots + x_1 - x_0 + x_0 = x_0 + z_1 + \cdots + z_n,$$

and

$$z_n = \mu Sz_{n-1} = \cdots = \mu^n S^n x_0,$$

that $\sum_n z_n$ is convergent, and we have

$$x = \lim_{n \rightarrow \infty} x_n = u + \mu Su + \mu^2 S^2 u + \cdots .$$

Example 1.29 *Solve*

$$x(t) - \mu \int_0^1 x(s) ds = 1$$

by means of the Neumann series, where we assume that $|\mu| < 1$. Try also to solve the equation directly.

In this case, $u(t) = 1$ and $k(t, s) = 1$, $a = 0$ and $b = 1$, thus $|\mu| < 1$ is a reasonable requirement (cf. EXAMPLE 1.28). It follows from EXAMPLE 1.28 that

$$x = 1 + \mu S + \mu^2 S^2 1 + \dots$$

We get from $S1 = \int_0^1 1 ds = 1$, that $S^2 1 = 1$. Then by induction, $S^n 1 = 1$, hence

$$x = 1 + \mu + \mu^2 + \dots = \frac{1}{1 - \mu}.$$

We now solve the equation directly. It follows from the rearrangement

$$x(t) = 1 + \mu \int_0^1 x(s) ds$$

that $x(t) = a$ must be a constant. Then by insertion,

$$a = 1 + \mu \cdot a,$$

hence

$$x(t) = a = \frac{1}{1 - \mu},$$

which apparently holds for every $\mu \neq 1$, and not just for $|\mu| < 1$.

2 Banach spaces

2.1 Simple vector spaces

Example 2.1 In the vector space $C([a, b])$ we consider the functions

$$e_0(t), e_1(t), \dots, e_n(t),$$

where $e_j(t)$ is a polynomial of degree j , where $j = 0, 1, \dots, n$,
 Show that e_0, e_1, \dots, e_n are linearly independent.

Since $e_0(t) = e_0 \neq 0$, we infer from $a_0 e_0 = 0$ that $a_0 = 0$, and the claim is true for $k = 0$.

First let $e_k(t) = t^k$, and assume that the claim is true for $k = 0, 1, \dots, n$. Now let

$$a_0 + a_1 t + \dots + a_n t^n + a_{n+1} t^{n+1} \equiv 0 \quad \text{for } t \in [a, b].$$

We get by a differentiation,

$$a_1 + 2a_2 t + \dots + na_n t^{n-1} + (n+1)a_{n+1} t^n \equiv 0 \quad \text{for } t \in [a, b],$$

thus $ka_k = 0, k = 1, 2, \dots, n+1$, according to the assumption of induction. We conclude that $a_k = 0$ for $k = 1, 2, \dots, n+1$, which by insertion gives the condition $a_0 = 0$. Then it follows by induction that $\{t^n \mid n \in \mathbb{N}_0\}$ are linearly independent.

Then let

$$e_k(t) = \sum_{j=0}^k e_{kj} t^j, \quad e_{kk} \neq 0,$$

and assume that

$$0 \equiv \sum_{k=0}^n a_k e_k(t) = \sum_{k=0}^n \sum_{j=0}^k a_k e_{kj} t^j = \sum_{j=0}^n \left\{ \sum_{k=j}^n a_k e_{kj} \right\} t^j.$$

It follows from the result above that

$$\sum_{k=j}^n a_k e_{kj} = 0 \quad \text{for } j = 0, 1, \dots, n.$$

We get for $j = n$ that $a_n e_{nn} = 0$, and since $e_{nn} \neq 0$, we must have $a_n = 0$. Since $e_{k,k+j} = 0$ for $j \geq 1$, the equation is reduced to

$$0 \equiv \sum_{j=0}^n \left\{ \sum_{k=j}^n a_k e_{kj} \right\} t^j = \sum_{j=0}^n \left\{ \sum_{k=j}^{n-1} a_k e_{kj} \right\} t^j = \sum_{j=0}^{n-1} \left\{ \sum_{k=j}^{n-1} a_k e_{kj} \right\} t^j,$$

where we as before conclude that $a_{n-1} = 0$. Then by recursion,

$$a_{n-2} = \dots = a_1 = a_0 = 0.$$

Example 2.2 Let U_1 and U_2 be subspaces of the vector space V . Show that $U_1 \cap U_2$ is a subspace. Is $U_1 \cup U_2$ always a subspace? If no, state conditions such that $U_1 \cup U_2$ is a subspace.

If U_1 and U_2 are subspaces, then

$$\forall \lambda \forall u, v \in U_i : u + \lambda v \in U_i, \quad i = 1, 2.$$

If $u, v \in U_1 \cap U_2$, then in particular, $u, v \in U_i, i = 1, 2$, thus $u + \lambda v \in U_i, i = 1, 2$, according to the above. It follows that $u + \lambda v \in U_1 \cap U_2$, hence $U_1 \cap U_2$ is also a subspace.

On the other hand, $U_1 \cup U_2$ is rarely a subspace. E.g. the X -axis and the Y -axis are two subspaces in \mathbb{R}^2 , and it is obvious that the union of the two axes is not a subspace.

The condition is that $U_1 \subseteq U_2$, or $U_1 \supseteq U_2$. In fact, if one of these conditions is satisfied, then it is obvious that $U_1 \cup U_2 = U_i$, where i is one of the numbers 1, 2.

If this condition is not fulfilled, then there exist

$$u_1 \in U_1 \setminus U_2 \quad \text{and} \quad u_2 \in U_2 \setminus U_1.$$

Assume that $u_1 + u_2 \in U_1 \cup U_2$, e.g. $u_1 + u_2 \in U_1$. Then $u_2 = (u_1 + u_2) - u_1 \in U_1$ contradicting the assumption. Hence we conclude that $u_1 + u_2 \notin U_1 \cup U_2$, and $U_1 \cup U_2$ is not a subspace.

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Example 2.3 Let V denote the set of all real $n \times n$ matrices.
 Show that V with the usual scalar multiplication and addition is a vector space.
 Is the set of all regular $n \times n$ -matrices a subspace of V ?
 Is the set of all symmetric $n \times n$ matrices a subspace of V ?

The first question is trivial: Since 0 is the zero element, and since 0 is not regular, the set of all regular matrices is not a subspace.

The set of all symmetric matrices is of course a subspace. In fact, if (a_{ij}) and (b_{ij}) are symmetric, thus $a_{ij} = a_{ji}$ and $b_{ij} = b_{ji}$, then

$$\lambda(a_{ij}) + (b_{ij}) = (\lambda a_{ij} + b_{ij}),$$

where

$$\lambda a_{ij} + b_{ij} = \lambda a_{ji} + b_{ji},$$

hence $(\lambda a_{ij} + b_{ij})$ is again symmetric.

Example 2.4 In the space $C([a, b])$ we consider the sets

$U_1 =$ the set of polynomials defined on $[a, b]$.

$U_2 =$ the set of polynomials defined on $[a, b]$ of degree $\leq n$.

$U_3 =$ the set of polynomials defined on $[a, b]$ of degree $= n$.

$U_4 =$ the set of all $f \in C([a, b])$ with $f(a) = f(b) = 0$.

$U_5 = C^1([a, b])$.

Which of the U_i , $i = 1, 2, \dots, 5$, are subspaces of $C([a, b])$?

$U_1 =$ the set of all polynomials is a subspace.

$U_2 =$ the set of all polynomials of degree $\leq n$ is a subspace.

$U_3 =$ the set of all polynomials of degree $= n$ is not a subspace. E.g. 0 does not belong to U_3 .

$U_4 =$ the set of all $f \in C^0([a, b])$ with $f(a) = f(b) = 0$ is a subspace.

$U_5 = C^1([a, b])$ is a subspace.

Example 2.5 In $C([-1, 1])$ we consider the sets U_1 and U_2 consisting of the odd and even functions in $C([-1, 1])$, respectively.

Show that U_1 and U_2 are subspaces and that $U_1 \cap U_2 = \{0\}$.

Show that every $f \in C([-1, 1])$ can be written in the form $f = f_1 + f_2$, where $f_1 \in U_1$ and $f_2 \in U_2$, and that this decomposition is unique.

If f, g are odd (even) functions, then $f + \lambda g$ is again an odd (even) function. Hence U_1 and U_2 are subspaces.

If $f \in U_1 \cap U_2$, then both

$$f(-t) = f(t) \quad \text{and} \quad f(-t) = -f(t),$$

thus $f(t) = -f(t)$ for all t , and we conclude that $2f(t) \equiv 0$. We conclude that $f \equiv 0$.

We see from

$$f(t) = \frac{f(t) + f(-t)}{2} + \frac{f(t) - f(-t)}{2},$$

where

$$\frac{f(t) + f(-t)}{2} \text{ is even, and } \frac{f(t) - f(-t)}{2} \text{ is odd,}$$

that such a splitting exists.

Assume that

$$f(t) = f_1(t) + f_2(t) = g_1(t) + g_2(t),$$

where f_1 and g_1 are odd, while f_2 and g_2 are even. Then

$$f_1(t) - g_1(t) = g_2(t) - f_2(t) \in U_1 \cap U_2 = \{0\},$$

hence $f_1 - g_1 = 0$ and $g_2 - f_2 = 0$. We conclude that $f_1 = g_1$ and $f_2 = g_2$, and the splitting is unique.

2.2 Normed spaces

Example 2.6 In the space $C^1([a, b])$ we have the norm

$$\|f\|_\infty = \sup_{t \in [a, b]} |f(t)|.$$

Show that we could take $\sup_{t \in (a, b)} |f(t)|$ instead.

Show that $C^1([a, b])$ with the sup-norm is not a Banach space.

Show that

$$\|f\|_\infty^* = \sup_{t \in [a, b]} |f(t)| + \sup_{t \in [a, b]} |f'(t)|$$

is also a norm on $C^1([a, b])$ and that it is a Banach space with this norm.

Every $f \in C^1([a, b])$ is continuous, so

$$\sup_{t \in [a, b]} |f(t)| = \sup_{t \in (a, b)} |f(t)|,$$

and we can use any of the two sup-norms.

It follows from Weierstraß's Approximation Theorem that the set \mathcal{P} of polynomials on $[a, b]$ is dense in $C^0([a, b])$ in the uniform norm. Since

$$\mathcal{P} \subset C^1([a, b]) \subset C^0([a, b])$$

and $C^1([a, b]) \neq C^0([a, b])$, we infer that $C^1([a, b])$ cannot be complete, thus $(C^1([a, b]), \|\cdot\|)$ is not a Banach space.

Then we shall prove that $\|\cdot\|_\infty^*$ is a norm.

1) Clearly, $\|f\|_\infty^* \geq 0$.

2) If

$$\|f\|_\infty^* = \sup_{t \in [a, b]} |f(t)| + \sup_{t \in [a, b]} |f'(t)| = \|f\|_\infty + \|f'\|_\infty = 0,$$

then in particular $\|f\| = 0$, so $f = 0$, because f is continuous.

3)

$$\|\lambda f\|_\infty^* = \|\lambda f\|_\infty + \|\lambda f'\|_\infty = |\lambda|(\|f\|_\infty + \|f'\|_\infty) = |\lambda| \cdot \|f\|_\infty^*.$$

4)

$$\begin{aligned} \|f + g\|_\infty^* &= \|f + g\|_\infty + \|f' + g'\|_\infty \leq \|f\|_\infty + \|g\|_\infty + \|f'\|_\infty + \|g'\|_\infty \\ &= (\|f\|_\infty + \|f'\|_\infty) + (\|g\|_\infty + \|g'\|_\infty) = \|f\|_\infty^* + \|g\|_\infty^*. \end{aligned}$$

We have proved that $\|\cdot\|_\infty^*$ is a norm on $C^1([a, b])$.

It “only” remains to prove that $(C^1([a, b]), \|\cdot\|_\infty^*)$ is a Banach space. Let (f_n) be a Cauchy sequence, i.d.

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} \forall m, n \in \mathbb{N} : m, n \geq N \implies \|f_m - f_n\|_\infty^* < \varepsilon.$$

It follows from $\|f\|_\infty^* = \|f\|_\infty + \|f'\|_\infty$, that $\|f\|_\infty \leq \|f\|_\infty^*$ and $\|f'\|_\infty \leq \|f\|_\infty^*$, thus (f_n) and (f'_n) are Cauchy sequences in the Banach space $(C^0([a, b]), \|\cdot\|_\infty)$. Hence there are continuous functions $f, g \in C^0([a, b])$, such that

$$f_n \rightarrow f \quad \text{and} \quad f'_n \rightarrow g.$$

Notice that it is not possible from this directly to conclude that

$$a) f \in C^1([a, b]), \quad b) f' = g.$$

A proof is required:

Define a function $h \in C^1([a, b])$ by

$$h(x) = \int_a^x g(t) dt + f(a), \quad x \in [a, b].$$

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We shall prove that $h(x) = f(x)$. It suffices to prove that $f_n \rightarrow h$ uniformly, because the limit function $f \in C^0([a, b])$ is unique. From $f_n \in C^1([a, b])$ follows that

$$f_n(x) = \int_a^x f'_n(t) dt + f_n(a), \quad x \in [a, b],$$

hence for every $x \in [a, b]$,

$$\begin{aligned} |f_n(x) - h(x)| &= \left| \int_a^x f'_n(t) dt + f_n(a) - \int_a^x g(t) dt - f(a) \right| \\ &\leq \left| \int_a^x \{f'_n(t) - g(t)\} dt \right| + |f_n(a) - f(a)|. \end{aligned}$$

Let $\varepsilon > 0$ be given. Since $f_n(a) \rightarrow f(a)$, and $f'_n \rightarrow g$ uniformly for $n \rightarrow +\infty$, there exists an $n_0 \in \mathbb{N}$, such that for every $n \geq n_0$,

$$|f_n(a) - f(a)| < \frac{\varepsilon}{2} \quad \text{and} \quad \sup_{t \in [a, b]} |f'_n(t) - g(t)| < \frac{\varepsilon}{2(b-a)}.$$

Therefore, if $n \geq n_0$, then for every $x \in [a, b]$,

$$|f_n(x) - h(x)| < \left| \int_a^x \frac{\varepsilon}{2(b-a)} dt \right| + \frac{\varepsilon}{2} \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

thus

$$\|f_n - h\|_\infty < \varepsilon \quad \text{for all } n \geq n_0,$$

and we have proved that $f_n \rightarrow h$ uniformly, hence $f = h$. Finally, since $h' = g$, the claim is proved.

Example 2.7 Let $f \in C([a, b])$ and consider the p -norms

$$\|f\|_p = \left\{ \int_a^b |f(t)|^p dt \right\}^{\frac{1}{p}}, \quad p \geq 1,$$

and

$$\|f\|_\infty = \sup_{t \in [a, b]} |f(t)|.$$

Show that $\|f\|_p \rightarrow \|f\|_\infty$ for $p \rightarrow \infty$.

The interval $[a, b]$ is bounded, so

$$\|f\|_p = \left\{ \int_a^b |f(t)|^p dt \right\}^{\frac{1}{p}} \leq \left\{ \int_a^b \|f\|_\infty^p dt \right\}^{\frac{1}{p}} = \|f\|_\infty (b-a)^{\frac{1}{p}}.$$

The function f is continuous and $[a, b]$ is compact, hence there exists a $t_0 \in [a, b]$, such that

$$|f(t_0)| = \|f\|_\infty.$$

To every $\varepsilon > 0$ we can find an interval $[c_\varepsilon, d_\varepsilon] \subseteq [a, b]$, $c_\varepsilon < d_\varepsilon$ (independently of p), such that

$$|f(t)| \geq (1 - \varepsilon)\|f\|_\infty \quad \text{for all } t \in [c_\varepsilon, d_\varepsilon].$$

Then we get the estimate

$$\begin{aligned} \|f\|_p &= \left\{ \int_a^b |f(t)|^p dt \right\}^{\frac{1}{p}} \geq \left\{ \int_{c_\varepsilon}^{d_\varepsilon} |f(t)|^p dt \right\}^{\frac{1}{p}} \geq \left\{ (1 - \varepsilon)^p \|f\|_\infty^p \int_{c_\varepsilon}^{d_\varepsilon} dt \right\}^{\frac{1}{p}} \\ &= (1 - \varepsilon)\|f\|_\infty \cdot (d_\varepsilon - c_\varepsilon)^{\frac{1}{p}}. \end{aligned}$$

Summing up we get for every $\varepsilon > 0$ that

$$(1 - \varepsilon)\|f\|_\infty \cdot (d_\varepsilon - c_\varepsilon)^{\frac{1}{p}} \leq \|f\|_p \leq \|f\|_\infty \cdot (b - a)^{\frac{1}{p}}.$$

If $k > 0$ is kept fixed, we have $k^{\frac{1}{p}} \rightarrow 1$ for $p \rightarrow \infty$. To every $\varepsilon > 0$ there exists a $P_\varepsilon > 0$, such that for every $p \geq P_\varepsilon$,

$$(d_\varepsilon - c_\varepsilon)^{\frac{1}{p}} \geq 1 - \varepsilon \quad \text{and} \quad (b - a)^{\frac{1}{p}} \leq 1 + \varepsilon,$$

hence

$$(1 - \varepsilon)^2 \|f\|_\infty \leq \|f\|_p \leq (1 + \varepsilon)\|f\|_\infty \quad \text{for every } p \geq P_\varepsilon.$$

This proves that $\lim_{p \rightarrow +\infty} \|f\|_p$ exists and that

$$\lim_{p \rightarrow +\infty} \|f\|_p = \|f\|_\infty.$$

Example 2.8 Let V be a normed vector space and let x_1, \dots, x_k be k linearly independent vectors from V . Show that there exists a positive constant m , such that for all scalars $\alpha_i \in \mathbb{C}$, $i = 1, \dots, k$, we have

$$\|\alpha_1 x_1 + \dots + \alpha_k x_k\| \geq m(|\alpha_1| + \dots + |\alpha_k|).$$

Indirect proof. We assume that there exists a sequence (y_m) , where

$$y_m = \sum_{i=1}^k \beta_i^{(m)} x_i, \quad \text{where } \sum_{i=1}^k |\beta_i^{(m)}| = 1 \text{ for all } m \in \mathbb{N},$$

and where $\|y_m\| \rightarrow 0$ for $m \rightarrow +\infty$. Under these assumptions we first notice that $|\beta_i^{(m)}| \leq 1$, such that $\left(\beta_i^{(m)}\right)_{m=1}^{+\infty}$ is a bounded sequence of complex numbers. The complex numbers \mathbb{C} being complete in the absolute value, there exists a convergent subsequence

$$\left(\beta_1^{(m_j)}\right)_{j=1}^{+\infty} \quad \text{af} \quad \left(\beta_1^{(m)}\right).$$

The trick is first to thin out $(\beta_2^{(m)})$ to the subsequence $\left(\beta_1^{(m_j)}\right)$, where (m_j) is given above.

Then thin it out once more to get a convergent subsequence

$$\left(\beta_2^{(m_j^2)}\right) \quad \text{of} \quad \left(\beta_2^{(m_j^1)}\right).$$

Because (m_j^2) is a subsequence of (m_j^1) , the subsequence $\left(\beta_1^{(m_j^2)}\right)$ is also convergent.

Continue in this way. After k steps we have obtained a subsequence (m_j) from \mathbb{N} , such that

$$\left(\beta_i^{(m_j)}\right)_{j=1}^{+\infty} \quad \text{is convergent for all } i = 1, 2, \dots, k.$$

This means that (y_{m_j}) is a convergent subsequence of (y_m) , hence

$$y_{m_j} \rightarrow y \quad \text{for } j \rightarrow +\infty,$$

and

$$y = \sum_{i=1}^k \beta_i x_i.$$

We conclude from

$$\sum_{i=1}^k |\beta_i| \geq \sum_{i=1}^k \left| \beta_i^{(m_j)} \right| - \sum_{i=1}^k \left| \beta_i^{(m_j)} - \beta_i \right| = 1 - \sum_{i=1}^k \left| \beta_i^{(m_j)} - \beta_i \right| \rightarrow 1, \quad \text{for } j \rightarrow +\infty,$$

and from the assumption that x_1, \dots, x_k are linearly independent that $y \neq 0$. This is contradicting the assumption that $\|y_m\| \rightarrow 0$ for $m \rightarrow +\infty$.

We infer that if $\sum_{i=1}^k |\beta_i| = 1$, then there is a constant $c > 0$, such that

$$\left\| \sum_{i=1}^k \beta_i x_i \right\| \geq c.$$

We put for $(\alpha_1, \dots, \alpha_k) \neq (0, \dots, 0)$,

$$\beta_i = \frac{\alpha_i}{|\alpha_1| + \dots + |\alpha_k|}.$$

Then the claim follows when we multiply by $|\alpha_1| + \dots + |\alpha_k| \neq 0$.

Finally, we notice that the case $\alpha_1 = \dots = \alpha_k = 0$ follows trivially for quite other reasons.

Example 2.9 Let V be a vector space and let $\|\cdot\|$ and $\|\|\cdot\|\|$ be two norms on V . The norms are said to be equivalent if there are positive constants m and M such that

$$m\|x\| \leq \|\|x\|\| \leq M\|x\|$$

for all $x \in V$.

Show that all norms on a finite dimensional vector space are equivalent.

Show that all equivalent norms define the same closed sets.

Let e_1, \dots, e_k be a basis for V . It follows from EXAMPLE 2.8 that there are constants $c_1 > 0$ and $c_2 > 0$, such that

$$\left\| \sum_{i=1}^k \alpha_i e_i \right\| \geq c_1 \sum_{i=1}^k |\alpha_i| \quad \text{and} \quad \|\| \sum_{i=1}^k \alpha_i e_i \|\| \geq c_2 \sum_{i=1}^k |\alpha_i|.$$

Writing $x = \sum_{i=1}^k \alpha_i e_i$, we get

$$\begin{aligned} \|x\| &= \left\| \sum_{i=1}^k \alpha_i e_i \right\| \leq \sum_{i=1}^k |\alpha_i| \cdot \|e_i\| \leq \max_{1 \leq i \leq k} \|e_i\| \cdot \sum_{j=1}^k |\alpha_j| \leq \frac{1}{c_2} \max_{1 \leq i \leq k} \|e_i\| \cdot \|\| \sum_{j=1}^k \alpha_j e_j \|\| \\ &= \frac{1}{c_2} \max_{1 \leq i \leq k} \|e_i\| \cdot \|x\| \leq \frac{1}{c_2} \max_{1 \leq i \leq k} \|e_i\| \cdot \sum_{j=1}^k |\alpha_j| \cdot \|e_j\| \\ &\leq \frac{1}{c_2} \max_{1 \leq i \leq k} \|e_i\| \cdot \max_{1 \leq j \leq k} \|e_j\| \cdot \sum_{\ell=1}^k |\alpha_\ell| \leq \frac{1}{c_1} \cdot \frac{1}{c_2} \max_{1 \leq i \leq k} \|e_i\| \cdot \max_{1 \leq j \leq k} \|e_j\| \cdot \|x\|. \end{aligned}$$

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Thus we have proved that

$$\|x\| \leq a \cdot \|x\| \leq b \cdot \|x\|,$$

where

$$a = \frac{1}{c_2} \max_{1 \leq i \leq k} \|e_i\| > 0 \quad \text{and} \quad b = a \cdot \frac{1}{c_1} \max_{1 \leq j \leq k} \|e_j\| > 0.$$

When we divide by $a > 0$, we get

$$m\|x\| = \frac{1}{a} \|x\| \leq \|x\| \leq \frac{b}{a} \|x\| = M\|x\|,$$

and we have proved that any two norms on a finite dimensional subspace are equivalent.

Since

$$m\|x\| \leq \|x\| \leq M\|x\|, \quad 0 < m \leq M,$$

and

$$\frac{1}{M} \|x\| \leq \|x\| \leq \frac{1}{m} \|x\|,$$

are equivalent, it suffices to prove that if U is closed with respect to $\|\cdot\|$, then U is also closed with respect to $\|\cdot\|$.

It is well-known (cf. EXAMPLE 1.10) that U is closed, if and only if

$$x_n \in U \text{ and } x_n \rightarrow x \implies x \in U.$$

Assume that U is closed with respect to $\|\cdot\|$, and let $(x_n) \subseteq U$ be a sequence for which

$$\|x_n\| \rightarrow 0 \quad \text{for } n \rightarrow +\infty,$$

thus (x_n) is convergent with respect to the norm $\|\cdot\|$. We shall prove that $x \in U$. However,

$$\|x_n - x\| \leq \frac{1}{m} \|x_n - x\| \rightarrow 0 \quad \text{for } n \rightarrow +\infty,$$

so also $x_n \rightarrow x$ with respect to the norm $\|\cdot\|$. It follows from the condition of EXAMPLE 1.10 (applied with respect to $\|\cdot\|$) that $x \in U$, and the claim is proved.

Example 2.10 *Show that a compact set in a normed vector space V is closed and bounded. If V is finite dimensional, show that a closed and bounded set is compact.*

Assume that U is compact in V , i.e. every sequence $(x_n) \subseteq U$ has a subsequence (y_n) , which converges towards an element y in U . We shall prove that U is closed and bounded.

Assume that $(x_n) \subseteq U$ is convergent in V , thus $x_n \rightarrow x \in V$. It follows from EXAMPLE 1.10 that U is closed, if we can prove that also $x \in U$.

According to the assumption there is a subsequence (y_n) of (x_n) , such that $y_n \rightarrow y \in U$. However, since $x_n \rightarrow x$, also $y_n \rightarrow x$, and since the limit value is unique in normed spaces, we conclude that $x = y \in U$, and it follows that U is closed.

Then we shall prove that if U is compact, then U is bounded. *Indirect proof.* Assume that U is unbounded. Let $x_1 \in U$ be arbitrarily chosen. There exists an $x_2 \in U$, such that

$$\|x_2\| \geq 1 + \|x_1\|.$$

Choose inductively a sequence $(x_n) \subseteq U$, such that

$$\|x_{n+1}\| \geq 1 + \|x_n\|.$$

Then note that if x_n and x_{n+p} , $p \in \mathbb{N}$ are any two elements, then

$$\|x_{n+p}\| \geq 1 + \|x_{n+p-1}\| \geq 2 + \|x_{n+p-2}\| \geq \cdots \geq p + \|x_n\|,$$

hence

$$\|x_{n+p} - x_n\| \geq \|x_{n+p}\| - \|x_n\| \geq 0 \geq 1 \quad \text{for alle } p \in \mathbb{N},$$

proving that no subsequence of (x_n) is convergent, and U is not compact.

We get by contraposition that if U is compact, then U is bounded.

Assume now that V is finite dimensional and that U is bounded and closed. Let e_1, \dots, e_k denote a basis for V , and let the constant $c > 0$ be chosen as in EXAMPLE 2.8, such that

$$\left\| \sum_{i=1}^k \alpha_i e_i \right\| \geq c(|\alpha_1| + \cdots + |\alpha_k|) = c \sum_{i=1}^k |\alpha_i|.$$

Let $x_n \in U$, $x_n = \sum_{i=1}^k \alpha_i^n e_i$, be any sequence. It follows from U being bounded that $\|x\| \leq B$ for every $x \in U$, i.e.

$$|\alpha_i| \leq \sum_{i=1}^k |\alpha_i| \leq \frac{1}{c} \left\| \sum_{i=1}^k \alpha_i e_i \right\| \leq \frac{B}{c}$$

for all $i = 1, \dots, k$. Hence the sequence $(\alpha_1^n)_n$ is bounded, and it has therefore a convergent subsequence $(\alpha_1^{n_j^1})$.

Since $(\alpha_2^{n_j^1})$ is a bounded sequence, it has a convergent subsequence $(\alpha_2^{n_j^2})$, etc..

After k steps we have found a sequence (n_j) , for which $(\alpha_i^{n_j})_j$ is convergent for $j \rightarrow +\infty$ for every $i = 1, \dots, k$, of limit value α_i .

Putting

$$y_j = \sum_{i=1}^k \alpha_i^{n_j} e_i,$$

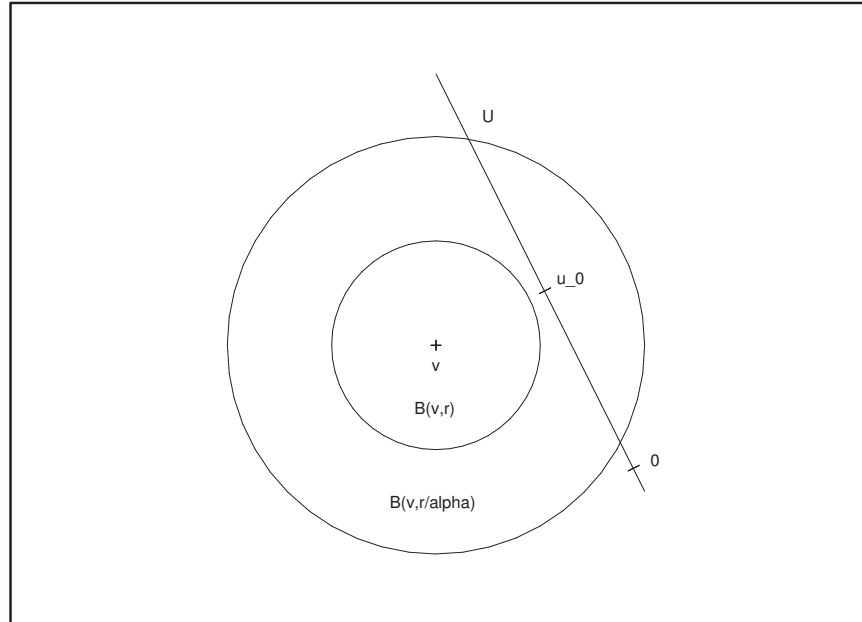
we get that (y_j) is convergent of limit

$$y_j \rightarrow y = \sum_{i=1}^k \alpha_i e_i.$$

Since $y_j \in U$, and U is closed, we get $y \in U$ according to EXAMPLE 1.10, and the claim is proved.

Example 2.11 *Riesz's lemma.* Let V be a normed vector space and let U be a closed subspace of V , $U \neq V$. Let α , $0 < \alpha < 1$, be given. Show that there is a $v \in V$, such that

$$\|v\| = 1 \quad \text{and} \quad \|v - u\| \geq \alpha \quad \text{for all } u \in U.$$



It follows from $U \neq V$, that there exists a $v \in V \setminus U$.

The set U is closed, so $V \setminus U$ is open. Hence there exists an $r > 0$, such that $B(v,r) \cap U = \emptyset$, where $B(v,r)$ denotes the open ball of centre v and radius r . This means that

$$(9) \quad \|v - u\| \geq r \quad \text{for all } u \in U.$$

Choose r sufficiently large such that (cf. the figure)

$$B(v,r) \cap U = \emptyset \quad \text{and} \quad B\left(v, \frac{1}{\alpha} r\right) \cap U \neq \emptyset.$$

Then for every $u_0 \in B\left(v, \frac{1}{\alpha} r\right) \cap U$,

$$(10) \quad r \leq \|v - u_0\| \leq \frac{1}{\alpha} r.$$

If we put

$$w = \frac{v - u_0}{\|v - u_0\|},$$

then $\|w\| = 1$.

We have for any $u \in U$ that

$$\|w - u\| = \left\| \frac{v - u_0}{\|v - u_0\|} - u \right\| = \frac{1}{\|v - u_0\|} \|v - u_0 - \|v - u_0\| u\|.$$

Now $u, u_0 \in U$, and U is a subspace, hence $u_0 + \|v - u_0\| u \in U$. By applying (9) with $u_0 + \|v - u_0\| u$ instead of u , it follows from (10) that

$$\|w - u\| = \frac{1}{\|v - u_0\|} \|v - (u_0 + \|v - u_0\| u)\| \geq \frac{r}{\|v - u_0\|} \geq \frac{r}{\frac{1}{\alpha} r} = \alpha.$$

We have proved that $w \in V$ satisfies

$$\|w\| = 1 \quad \text{and} \quad \|w - u\| \geq \alpha \quad \text{for every } u \in U.$$

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Example 2.12 In ℓ^∞ , the vector space of bounded sequences, we consider the sets U_1 and U_2 , where U_1 denotes the set of sequences with only finitely many elements different from 0 and U_2 the set of sequences with all but the N first elements equal to 0.

Are U_1 and/or U_2 closed subspaces in ℓ^∞ ?

Are U_1 and/or U_2 finite dimensional?

It follows from

$$x_n = \left(1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}, 0, 0, \dots\right) \in U_1,$$

and

$$x_n \rightarrow \left(\frac{1}{k}\right)_{k \in \mathbb{N}} \notin U_1,$$

that U_1 is not closed.

Of course U_1 is a subspace, and since every *finite dimensional* subspace is closed (which U_1 is not), we conclude that U_1 is not finite dimensional.

On the other hand, U_2 and \mathbb{R}^N are isomorphic, so U_2 is a closed and finite dimensional vector space, $\dim U_2 = N$.

Example 2.13 Let $(V, \|\cdot\|)$ be a normed vector space, and let U be the unit ball,

$$U = \{x \in V \mid \|x\| \leq 1\}.$$

Prove that U is compact, if and only if V is finite dimensional.

Obviously, U is closed and bounded. If V is finite dimensional, then it follows from EXAMPLE 2.10 that U is compact. It remains to be proved that if U is compact, then V is finite dimensional.

INDIRECT PROOF. Assume that V is not finite dimensional. Choose any $x_1 \in U$, such that $\|x_1\| = 1$. Then x_1 generates a subspace V_1 . Then by Riesz's lemma (EXAMPLE 2.11) there exists an $x_2 \in U$, such that

$$\|x_2\| = 1 \quad \text{and} \quad \|x_2 - \lambda x_1\| \geq \frac{1}{2} \quad \text{for all } \lambda.$$

By induction, using Riesz's lemma in each step, we obtain a sequence $x_n \in U$ of unit vectors, $\|x_n\| = 1$, such that

$$\left\|x_n - \sum_{j=1}^{n-1} \lambda_j x_j\right\| \geq \frac{1}{2} \quad \text{for any } \lambda_j.$$

We have in particular,

$$\|x_n - x_m\| \geq \frac{1}{2} \quad \text{for } n \neq m,$$

proving that (x_n) does not contain any convergent subsequence. Hence U is not compact.

We get by contraposition that if the unit ball U is compact, then the vector space V is finite dimensional.

Example 2.14 Consider in ℓ^p (where $1 \leq p \leq +\infty$) the subspace U consisting of all sequences which are 0 eventually.

- 1) If $1 \leq p < +\infty$, is the subspace U then dense in ℓ^p ?
- 2) If $p = +\infty$, is the subspace U then dense in ℓ^∞ ?

1) The answer is 'yes'. In fact, if $(x_j)_{j \in \mathbb{N}} \in \ell^p$, then

$$\sum_{j=1}^{+\infty} |x_j|^p < +\infty.$$

To every $\varepsilon > 0$ there is an N , such that

$$\sum_{j=N+1}^{+\infty} |x_j|^p < \varepsilon^p.$$

Putting $x^N = (x_1, \dots, x_N, 0, 0, \dots) \in U$, we get

$$\|x - x^N\|_p = \left\{ \sum_{j=N+1}^{+\infty} |x_j|^p \right\}^{\frac{1}{p}} < \{\varepsilon^p\}^{\frac{1}{p}} = \varepsilon.$$

2) In this case the answer is 'no'. In fact, if $x = (1, 1, 1, \dots) \in \ell^\infty$, then

$$\|x - y\|_\infty \geq 1 \quad \text{for every } y \in U.$$

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Example 2.15 On $C([a, b])$ we introduce the norm

$$\|f\|_p = \left\{ \int_a^b |f(t)|^p dt \right\}^{\frac{1}{p}}, \quad p \in]1, +\infty[.$$

Let $g \in C([a, b])$, and let q be given by $\frac{1}{p} + \frac{1}{q} = 1$. Prove that we by

$$T_g f = \int_a^b f(t) \overline{g(t)} dt$$

define a linear functional on $C([a, b])$, and that

$$\|T_g\| = \|g\|_q \quad \left(= \left\{ \int_a^b |g(t)|^q dt \right\}^{\frac{1}{q}} \right).$$

Most of the claims have already been proved, included the estimate $\|T_g\| \leq \|g\|_1$. We shall only proof that we even get equality. The trick is to choose a suitable $f \in C([a, b])$. We have

$$T_g f = \int_a^b f(t) \overline{g(t)} dt.$$

Since $g(t)$ is continuous, we get

$$g(t) = e^{i\varphi(t)} |g(t)|,$$

where $\varphi(t)$ can be chosen continuous in every interval, in which $g(t) \neq 0$.

Choosing

$$f(t) = e^{i\varphi(t)} |g(t)|^{\frac{q}{p}},$$

f is again continuous and

$$\|f\|_p^p = \int_a^b |g(t)|^q dt = \|g\|_q^q, \quad \text{thus} \quad \|f\|_p = \|g\|_q^{\frac{q}{p}} = \|g\|_q^{q-1},$$

and

$$\begin{aligned} T_g f &= \int_a^b f(t) \overline{g(t)} dt = \int_a^b e^{i\varphi(t)} |g(t)|^{\frac{q}{p}} e^{-i\varphi(t)} |g(t)| dt \\ &= \int_a^b |g(t)|^{\frac{q}{p}+1} dt = \int_a^b |g(t)|^{q(\frac{1}{p}+\frac{1}{q})} dt = \int_a^b |g(t)|^q dt \\ &= \|g\|_q^q = \|g\|_q \cdot \|g\|_q^{q-1} = \|g\|_q \cdot \|f\|_p. \end{aligned}$$

It follows from

$$|T_g f| = T_g f = \|g\|_q \|f\|_p \leq \|T_g\| \cdot \|f\|_p,$$

that $\|g\|_q \leq \|T_g\|$. Since already $\|T_g\| \leq \|g\|_q$, we must have $\|T_g\| = \|g\|_q$.

2.3 Banach spaces

Example 2.16 Show that a closed subspace of a Banach space is itself a Banach space.

Let U be a closed subspace of a Banach space V . Since V is complete, it follows from EXAMPLE 1.10 that U is also complete, hence U is a Banach space.

Example 2.17 Let $V_i, i = 1, 2, \dots, n$, be normed vector spaces, with norms $\|\cdot\|_i, i = 1, 2, \dots, n$. The product space $V_1 \times V_2 \times \dots \times V_n = \bigotimes_{i=1}^n V_i$ is defined by

$$\bigotimes_{i=1}^n V_i = \{(x_1, x_2, \dots, x_n) \mid x_i \in V_i, i = 1, 2, \dots, n\}.$$

In $\bigotimes_{i=1}^n V_i$ we use coordinate wise addition:

$$(x_1, x_2, \dots, x_n) + (y_1, y_2, \dots, y_n) = (x_1 + y_1, x_2 + y_2, \dots, x_n + y_n),$$

and scalar multiplication:

$$\lambda(x_1, x_2, \dots, x_n) = (\lambda x_1, \lambda x_2, \dots, \lambda x_n),$$

and we define the norm by

$$\|(x_1, x_2, \dots, x_n)\| = \sum_{i=1}^n \|x_i\|_i.$$

Show that $\bigotimes_{i=1}^n V_i$ with this norm is a normed vector space, and show that if all the spaces V_i with their respective norms are Banach spaces, then $\bigotimes_{i=1}^n V_i$ is a Banach space.

We shall prove the claim by induction over n . For $n = 1$ there is nothing to prove.

If $n = 2$, then clearly $V_1 \times V_2$ is a vector space with the operations addition and scalar multiplication defined above. Then we shall prove that

$$\|(x_1, x_2)\| = \|x_1\|_1 + \|x_2\|_2$$

is a norm.

Clearly, $\|(x_1, x_2)\| \geq 0$, and if $\|(x_1, x_2)\| = \|x_1\|_1 + \|x_2\|_2 = 0$, then both $\|x_1\|_1 = 0$ and $\|x_2\|_2 = 0$, thus $x_1 = 0$ og $x_2 = 0$.

Furthermore,

$$\|\lambda(x_1, x_2)\| = \|(\lambda x_1, \lambda x_2)\| = \|\lambda x_1\|_1 + \|\lambda x_2\|_2 = |\lambda| (\|x_1\|_1 + \|x_2\|_2) = |\lambda| \cdot \|(x_1, x_2)\|.$$

Finally,

$$\begin{aligned} \|(x_1, x_2) + (y_1, y_2)\| &= \|(x_1 + y_1, x_2 + y_2)\| = \|x_1 + y_1\|_1 + \|x_2 + y_2\|_2 \\ &\leq \|x_1\|_1 + \|y_1\|_1 + \|x_2\|_2 + \|y_2\|_2 \\ &= (\|x_1\|_1 + \|x_2\|_2) + (\|y_1\|_1 + \|y_2\|_2) \\ &= \|(x_1, x_2)\| + \|(y_1, y_2)\|, \end{aligned}$$

and we have proved that $\|\cdot\|$ is a norm on $V_1 \times V_2$.

Then assume that both V_1 and V_2 are complete, and let $((x_1^n, x_2^n))_n$ be a Cauchy sequence on $V_1 \times V_2$. It follows from

$$\|x_i^n - x_i^m\|_i \leq \|(x_1^n - x_1^m, x_2^n - x_2^m)\| = \|(x_1^n, x_2^n) - (x_1^m, x_2^m)\|, \quad t = 1, 2,$$

that $(x_i^n)_n$ are Cauchy sequences on V_i , $i = 1, 2$, hence convergent with limit values x_i , $i = 1, 2$. By this construction we then get

$$\|(x_1, x_2) - (x_1^n, x_2^n)\| = \|x_1 - x_1^n\|_1 + \|x_2 - x_2^n\|_2 \rightarrow 0 \quad \text{for } n \rightarrow +\infty,$$

proving that $(x_1^n, x_2^n) \rightarrow (x_1, x_2) \in V_1 \times V_2$. We have proved that $V_1 \times V_2$ is complete, thus $(V_1 \times V_2, \|\cdot\|)$ is a Banach space.

Assume that the claims are true for some $n \in \mathbb{N}$ (this is true by the above for $n = 1$ and for $n = 2$), and consider $\bigotimes_{i=1}^{n+1} U_i$, where each U_i is a normed vector space (a Banach space). We define

$$V_1 = \bigotimes_{i=1}^n U_i \quad \text{and} \quad V_2 = U_{n+1}.$$

It follows from the assumption of the induction that $(V_1, \|\cdot\|_n^*)$ is a normed vector space (or a Banach space) under the given assumptions, and the same is true for the space $(V_2, \|\cdot\|_{n+1})$. It only remains to notice that

$$\|(x_1, x_2, \dots, x_n)\|_n^* = \|x_1\|_1 + \|x_2\|_2 + \dots + \|x_n\|_n,$$

hence

$$\|(x_1, \dots, x_n, x_{n+1})\| = \|(x_1, \dots, x_n)\|_n^* + \|x_{n+1}\|_{n+1}.$$

It follows that $\bigoplus_{i=1}^{n+1} U_i$ is a normed vector space (or a Banach space) under the given assumptions.

Example 2.18 Assume that V and U are normed spaces and $f : V \rightarrow U$ is a continuous mapping, and assume that $X \subset V$ is a compact subset. Show that the image $f(X) \subset U$ is compact. Show that a real function attains both maximum and minimum on a compact set.

There are several definitions of compactness. We shall here use *sequential compactness*, which is defined by X being sequential compact, if every sequence on X has a convergent subsequence.

We shall prove that if $f : V \rightarrow U$ is continuous, and $X \subset V$ is compact, then the image $f(X) \subset U$ is also compact.

Let $(y_n) \subset f(X)$ be any sequence on the image $f(X)$. There exists a sequence $(x_n) \subset X$, such that $y_n = f(x_n)$ for every $n \in \mathbb{N}$. Since X is compact, (x_n) has a convergent subsequence $(x'_n) \subseteq (x_n)$, where $x'_n \rightarrow x_0 \in X$ for $n \rightarrow +\infty$.

Now, f is continuous at $x_0 \in X$, so to every $\varepsilon > 0$ there exists a $\delta > 0$, such that

$$\|f(x'_n) - f(x_0)\|_U < \varepsilon \quad \text{for } \|x'_n - x_0\|_V < \delta.$$

Then $(x'_n) \rightarrow x_0$ implies that there exists an $n_0 \in \mathbb{N}$, such that

$$\|x'_n - x_0\|_V < \delta \quad \text{for all } n \geq n_0.$$

We have for the same n_0 that

$$\|f(x'_n) - f(x_0)\|_U < \varepsilon \quad \text{for all } n \geq n_0,$$

which means that $(f(x'_n))$ converges towards $f(x_0)$, thus every sequence $(y_n) = (f(x_n)) \subseteq f(X)$ has a convergent subsequence $(y'_n) = (f(x'_n))$. Note for the limit point that $f(x_0) \in f(X)$.

Assume that $f : X \rightarrow \mathbb{R}$ is continuous, where X is a compact subset of a normed space. It follows from the above that $f(X) \subseteq \mathbb{R}$ is compact, thus closed and bounded in \mathbb{R} . In particular, f has both a maximum value and a minimum value.

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Example 2.19 Show that any finite dimensional subspace of a normed vector space is a Banach space.

Let $(V, \|\cdot\|)$ be the normed space, and let U be a finite dimensional subspace of V . Let e_1, \dots, e_k , denote a basis for U . It follows from EXAMPLE 2.8 that there exists a constant $c > 0$ (corresponding to the basis e_1, \dots, e_k), such that

$$\left\| \sum_{i=1}^k \alpha_i e_i \right\| \geq c(|\alpha_1| + \dots + |\alpha_k|).$$

Let $x^n = \sum_{i=1}^k \alpha_i^n e_i$ denote a Cauchy sequence on U , thus

$$\forall \varepsilon > 0 \exists N \forall m, n \geq N : \|x^m - x^n\| = \left\| \sum_{i=1}^k (\alpha_i^m - \alpha_i^n) e_i \right\| < \varepsilon.$$

Then in particular,

$$|\alpha_i^m - \alpha_i^n| \leq \sum_{i=1}^k |\alpha_i^m - \alpha_i^n| \leq \frac{1}{c} \left\| \sum_{i=1}^k (\alpha_i^m - \alpha_i^n) e_i \right\| < \frac{\varepsilon}{c} \quad \text{for } m, n \geq N.$$

It follows that $(\alpha_i^n)_n$ is a Cauchy sequence on \mathbb{C} for every $i = 1, \dots, k$, hence convergent, $\alpha_i^n \rightarrow \alpha_i$ for $n \rightarrow +\infty$.

In this way we construct an element

$$x = \sum_{i=1}^k \alpha_i e_i \in U.$$

It remains to be proved that $x^n \rightarrow x$ for $n \rightarrow +\infty$. However,

$$\|x - x^n\| = \left\| \sum_{i=1}^k (\alpha_i - \alpha_i^n) e_i \right\| \leq \sum_{i=1}^k |\alpha_i - \alpha_i^n| \cdot \|e_i\| \rightarrow 0 \quad \text{for } n \rightarrow +\infty,$$

because every term in the finite sum tends towards 0 for $n \rightarrow +\infty$. This proves that every finite dimensional subspace of a normed vector space is a Banach space.

Example 2.20 Let V be a Banach space. A series $\sum_{k=0}^{\infty} x_k$, $x_k \in V$, is convergent if the sequence (s_n) , where

$$s_n = \sum_{k=0}^n x_k,$$

is convergent in V .

Show that $\sum_{k=0}^{\infty} \|x_k\| < \infty$ implies that $\sum_{k=0}^{\infty} x_k$ is convergent.

Does the convergence of $\sum_{k=0}^{\infty} x_k$ imply that $\sum_{k=0}^{\infty} \|x_k\| < \infty$?

What if the space V is only assumed to be a normed space?

1) Given a Banach space V . It suffices to prove that (s_n) is a Cauchy sequence.

Let $\varepsilon > 0$ be given. Since

$$\sum_{k=0}^{\infty} \|x_k\| < +\infty,$$

is finite, there exists an N , such that

$$\sum_{k=N}^{\infty} \|x_k\| < \varepsilon.$$

It holds for $n > m \geq N$ that

$$\|s_n - s_m\| = \left\| \sum_{k=0}^n x_k - \sum_{k=0}^m x_k \right\| = \left\| \sum_{k=m+1}^n x_k \right\| \leq \sum_{k=m+1}^n \|x_k\| \leq \sum_{k=N}^{\infty} \|x_k\| < \varepsilon,$$

thus (s_n) is a Cauchy sequence in a Banach space, hence also convergent.

2) It is well-known that the claim does not hold in the simplest possible Banach space $(\mathbb{R}, |\cdot|)$, because there exist conditional convergent series like e.g.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \ln 2,$$

which are not absolutely convergent,

$$\sum_{n=1}^{\infty} \frac{1}{n} = \infty.$$

3) This is not true, either. Denote by c the vector space consisting of real sequences (x_n) , where $x_n = 0$ eventually, e.g. for $n \geq N(x)$. Choose as norm,

$$\|x\| = \sqrt{\sum_{n=1}^{\infty} x_n^2}.$$

Then c is dense in ℓ^2 , and $c \neq \ell^2$.

Choose $x_n = \frac{1}{n} e_n$. Then

$$\left\| \sum_{n=1}^{\infty} x_n \right\|^2 = \sum_{n=1}^{\infty} \|x_n\|^2 = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6},$$

so $\sum_{n=1}^{\infty} x_n \in \ell^2$.

Clearly, $\sum_{n=1}^{\infty} x_n$ is not zero, eventually, while all $s_n = \sum_{k=1}^n x_k$ have this property. Hence

$$c \ni s_n \rightarrow \sum_{n=1}^{\infty} x_n \in \ell^2 \setminus c.$$

Example 2.21 Let $(V, \|\cdot\|)$ denote a normed space. Let V' denote the set of all bounded linear functionals on $(V, \|\cdot\|)$. The set V' is organized as a vector space by the operations

$$(f + g)(x) = f(x) + g(x), \quad \text{for all } x \in V,$$

$$(\alpha f)(x) = \alpha f(x), \quad \text{for all } x \in V,$$

and we introduce a norm on V' by

$$\|f\|' = \sum_{\|x\| \leq 1} |f(x)|.$$

Prove that $(V', \|\cdot\|')$ is a Banach space. It is called the dual space V .

We shall first show that $\|\cdot\|'$ is a norm on V' . It is obvious that $\|f\|' \geq 0$. If $\|f\|' = 0$, then

$$\sup_{\|x\| \leq 1} |f(x)| = 0.$$

Then we have $\left\| \frac{x}{\|x\|} \right\| = 1$ for arbitrary $x \neq 0$, hence

$$|f(x)| = \left| f \left(\|x\| \cdot \frac{x}{\|x\|} \right) \right| = \|x\| \cdot \left| f \left(\frac{x}{\|x\|} \right) \right| = 0.$$

It follows from $f(0) = 0$ that $f(x) = 0$ for every $x \in V$, thus $f \equiv 0$. Furthermore,

$$\|\alpha f\|' = \sup_{\|x\| \leq 1} |\alpha f(x)| = |\alpha| \cdot \sup_{\|x\| \leq 1} |f(x)| = |\alpha| \cdot \|f\|',$$

and finally,

$$\begin{aligned} \|f + g\|' &= \sup_{\|x\| \leq 1} |f(x) + g(x)| \leq \sup_{\|x\| \leq 1} (|f(x)| + |g(x)|) \\ &\leq \sup_{\|x\| \leq 1} |f(x)| + \sup_{\|x\| \leq 1} |g(x)| = \|f\|' + \|g\|', \end{aligned}$$

and we have proved that $\|\cdot\|'$ is a norm.

Assume that (f_n) is a Cauchy sequence on V' , i.e.

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} \forall m, n \geq N : \|f_n - f_m\| < \varepsilon.$$

This means that

$$\|f_n - f_m\|' = \sup_{\|x\| \leq 1} |f_n(x) - f_m(x)| < \varepsilon \quad \text{for all } m, n \geq N,$$

i.e. we have for every x , for which $\|x\| \leq 1$ that $(f_n(x))$ is a Cauchy sequence in \mathbb{C} , hence convergent.

For any $x \neq 0$ it follows that $\frac{x}{\|x\|}$ is a unit vector, thus

$$\forall \varepsilon > 0 \exists N_x \in \mathbb{N} \forall m, n \geq N_x : \|f_n - f_m\|' < \frac{\varepsilon}{\|x\|},$$

which only means that

$$|f_n(x) - f_m(x)| = \|x\| \cdot \left| f_n\left(\frac{x}{\|x\|}\right) - f_m\left(\frac{x}{\|x\|}\right) \right| < \|x\| \cdot \frac{\varepsilon}{\|x\|} = \varepsilon,$$

so $(f_n(x))$ is convergent for every $x \in V \setminus \{0\}$. If $x = 0$, we just get $f_n(0) = 0 \rightarrow 0$ for $n \rightarrow +\infty$. If we put

$$f(x) = \lim_{n \rightarrow +\infty} f_n(x),$$

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then we have defined a functional on V for which in particular $f(0) = 0$. It remains only to prove that 1) f is linear, and at 2) f is bounded. However,

$$f(x + \lambda y) = \lim_{n \rightarrow +\infty} f_n(x + \lambda y) = \lim_{n \rightarrow +\infty} \{f_n(x) + \lambda f_n(y)\} = f(x) + \lambda f(y),$$

proving the linearity. Then

$$\begin{aligned} (11) \|f\|' &= \sup_{\|x\| \leq 1} |f(x)| = \sup_{\|x\| \leq 1} |f(x) - f_n(x) + f_n(x)| \\ &\leq \sup_{\|x\| \leq 1} |f(x) - f_n(x)| + \sup_{\|x\| \leq 1} |f_n(x)| \\ &= \sup_{\|x\| \leq 1} |f(x) - f_n(x)| + \|f_n\|'. \end{aligned}$$

Choose n , such that for all $m \geq n$,

$$\|f_n - f_m\|' = \sup_{\|x\| \leq 1} |f_n(x) - f_m(x)| < 1.$$

Then $f_m(x) \in B(f_n(x), 1)$ for every x , for which $\|x\| \leq 1$. Since $f_m(x) \rightarrow f(x)$ for $m \rightarrow +\infty$, we have $f(x) \in \overline{B(f_n(x), 1)}$, so $|f_n(x) - f(x)| \leq 1$ for all x , for which $\|x\| \leq 1$. From this we infer that

$$\sup_{\|x\| \leq 1} |f(x) - f_n(x)| \leq 1.$$

Therefore, if n is chosen as above, then it follows from (11) that $\|f\|' \leq 1 + \|f_n\|'$, hence f is bounded, and we have proved that every Cauchy sequence on V' is convergent, i.e. V' is a Banach space.

2.4 The Lebesgue integral

‘n

Example 2.22 Let $f \in L^1(\mathbb{R})$.

- 1) Can we conclude that $f(x) \rightarrow 0$ for $|x| \rightarrow \infty$?
- 2) Can we find $a, b \in \mathbb{R}$ such that $|f(x)| \leq b$ for $|x| \geq a$?

In both cases the answer is ‘no’. For example, $g(x) = x \cdot 1_{\mathbb{Z}}(x)$ fulfils none of the conditions, and

$$\int |g(x)| dx = 0.$$

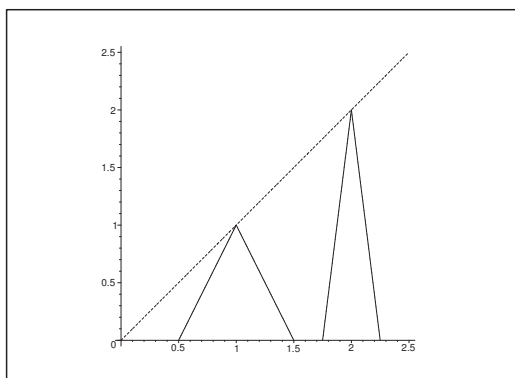


Figure 4: The graph of a *continuous function* $f(x)$, which does not fulfil the two requirements.

We shall now construct a function f , which is *continuous* and Lebesgue integrable, and which does not fulfil any of the two requirements above. Let

$$f(x) = \begin{cases} n & \text{for } x = n, & n \in \mathbb{N}, \\ 0 & \text{for } x = n \pm 2^{-n}, & n \in \mathbb{N}, \\ \text{piecewise linear,} & \text{otherwise.} \end{cases}$$

Clearly, f is continuous and satisfies neither (1) nor (2). We shall only prove that f is integrable. Now, $f \geq 0$, so

$$\int_{-\infty}^{+\infty} f(x) dx = \sum_{n=1}^{+\infty} \frac{1}{2} n \cdot 2 \cdot 2^{-n} = \sum_{n=1}^{+\infty} n 2^{-n} < +\infty,$$

and the claim is proved.

Remark 2.1 For completeness we here add the full proof. We have

$$\begin{aligned} \sum_{n=1}^{+\infty} n \cdot 2^{-n} &= 2 \sum_{n=1}^{+\infty} n \cdot 2^{-(n+1)} = 2 \sum_{n=2}^{+\infty} (n-1)2^{-n} = 2 \sum_{n=2}^{+\infty} n \cdot 2^{-n} - 2 \sum_{n=2}^{+\infty} 2^{-n} \\ &= 2 \sum_{n=1}^{+\infty} n \cdot 2^{-n} - 2 \cdot 1 \cdot 2^{-1} - \sum_{n=1}^{+\infty} 2^{-n} = 2 \sum_{n=1}^{+\infty} n \cdot 2^{-n} - 1 - 1, \end{aligned}$$

hence by a rearrangement,

$$\sum_{n=1}^{+\infty} n \cdot 2^{-n} = 2.$$

ALTERNATIVELY one may exploit that

$$\frac{d}{dz} \left(\frac{1}{1-z} \right) = \frac{1}{(1-z)^2} = \frac{d}{dz} \left(\sum_{n=0}^{+\infty} z^n \right) = \sum_{n=1}^{+\infty} n z^{n-1},$$

for $|z| < 1$. When we insert $z = \frac{1}{2}$, we easily get the result. \diamond

Example 2.23 *Prove that if $f : \mathbb{R} \rightarrow \mathbb{R}$ is monotonous, then f has at most countably many points of discontinuity.*

We may assume that f is increasing, thus $f(x) \geq f(y)$ for $x > y$. We may even restrict ourselves to the interval $[0, 1]$, because the number of intervals of the form $[n, n+1]$, $n \in \mathbb{Z}$, is countable. This means that we may assume that $f(x) = 0$ for $x \leq 0$, and $f(x) = 1$ for $x \geq 1$.

Let $\{x_j \mid j \in J\}$ be the set of all points of discontinuity in $[0, 1]$. Then to any x_j we can find an interval I_j with interior points on the Y -axis, such that $f(x) \notin I_j$ for all $x \in [0, 1]$, i.e. one jumps over the values in I_j over.

Every I_j can be “numbered” by a rational number $q_j \in I_j$, because \mathbb{Q} is dense in \mathbb{R} . This means that $\{x_j \mid j \in J\}$ contains just as many elements, as there are different elements in

$$\{q_j \mid j \in J\} \subseteq \mathbb{Q}.$$

Now, \mathbb{Q} is countable, so $\{q_j \mid j \in J\}$ is countable, and thus $\{x_j \mid j \in J\}$ is at most countable.

Define

$$f(x) = 2^{-n+1} \quad \text{for } x \in \left] \frac{1}{n+1}, \frac{1}{n} \right], \quad n \in \mathbb{N}.$$

Then f is monotonous of the countably many points of discontinuity $\left\{ \frac{1}{n} \mid n \in \mathbb{N} \setminus \{1\} \right\}$, showing that there exist monotonous functions with a countable number of points of discontinuity.

An ALTERNATIVE proof is the following: We may as before assume that f is increasing on the interval $[0, 1]$ with $f(x) = 0$ for $x \leq 0$ and $f(x) = 1$ for $x \geq 1$.

If x_0 is a point of discontinuity, then $f(x) \leq f(x_0)$ for every $x \leq x_0$. Hence, if $x_n \nearrow x_0$, then $(f(x_n))$ is an increasing bounded sequence of numbers, so $(f(x_n))$ is convergent with the limit value c .

Let $y_n \nearrow x_0$ be another such sequence of numbers. Then $(f(y_n)) \rightarrow c'$. We shall prove that $c = c'$. This is done INDIRECTLY.

Assume (e.g.) that $c < c'$, and let $0 < \varepsilon < c' - c$. Corresponding to this ε there exists an N , such that

$$|c' - f(y_n)| = c' - f(y_n) < \varepsilon \quad \text{for all } n \geq N.$$

To any y_n we can find an x_m , such that $y_n < x_m < x_0$, hence

$$f(y_n) \leq f(x_m) \quad [\leq c].$$

Then it follows that

$$\varepsilon < |c' - c| = c' - c = c' - f(y_n) + f(y_n) - c < \varepsilon + f(y_n) - c,$$

so $f(y_n) - c > 0$, and we have come to the contradiction

$$c < f(y_n) \leq f(x_m) \leq c \quad \text{for } n \geq N.$$

We therefore conclude that $c' = c$.

Since the limit value is the same, no matter how $x_n \nearrow x_0$ is chosen, we conclude that

$$c = \lim_{x \rightarrow x_0^-} f(x).$$

We prove in a similar way that $\lim_{x \rightarrow x_0^+} f(x)$ exists, and that these two values are different at any point of discontinuity.

Define the jump at a point of discontinuity x_0 as

$$\sigma_0 = \lim_{x \rightarrow x_0^+} f(x) - \lim_{x \rightarrow x_0^-} f(x) > 0.$$

If $x_0 < x_1$ are both points of discontinuity, then it follows from that the function is monotonous that

$$\lim_{x \rightarrow x_0^+} f(x) \leq \lim_{x \rightarrow x_1^-} f(x).$$

Let $\{x_j \mid j \in J\}$ denote the set of point of discontinuity in $[0, 1]$. The image is contained in $[0, 1]$, hence

$$\sum_{x_j} \sigma_j \leq 1,$$

and the sum is finite. Every $\sigma_j > 0$, so the sum is at most countable, thus $J \subseteq \mathbb{N}$, and the claim is proved.

Example 2.24 Prove that $f(x) = \frac{|\sin x|}{x}$ is not Lebesgue integrable on $[\pi, +\infty[$, thus $f \notin L^1([\pi, +\infty[)$.

HINT: Consider

$$f_n(x) = \begin{cases} \frac{|\sin x|}{x}, & \pi \leq x \leq n\pi, \\ 0, & \text{otherwise,} \end{cases}$$

and exploit that $f_n(x) \nearrow f(x)$ and $\int_{\pi}^{\infty} f_n(x) dx \geq \frac{1}{3} \sum_{k=2}^n \frac{1}{k}$.

Let f_n be given as above. Then clearly,

$$0 \leq f_n(x) \nearrow f(x).$$

Furthermore,

$$\begin{aligned} \int_{\pi}^{\infty} f_n(x) dx &= \int_{\pi}^{n\pi} \frac{|\sin x|}{x} dx = \sum_{k=2}^n \int_{(k-1)\pi}^{k\pi} \frac{|\sin x|}{x} dx \\ &\geq \sum_{k=2}^n \int_{(k-1)\pi}^{k\pi} \frac{1}{k\pi} \cdot |\sin x| dx = \sum_{k=2}^n \frac{1}{k\pi} \left| \int_{(k-1)\pi}^{k\pi} \sin x dx \right| \\ &= \sum_{k=2}^n \frac{1}{k\pi} \left| \int_0^{\pi} \sin x dx \right| = \sum_{k=2}^n \frac{2}{k\pi} = \frac{2}{\pi} \sum_{k=2}^n \frac{1}{k} \rightarrow +\infty \quad \text{for } n \rightarrow +\infty, \end{aligned}$$

and we infer that f is not Lebesgue integrable, i.e. f does not belong to $L^1([\pi, \infty[)$.

Example 2.25 Give a simple proof of Hölder's inequality in the case of $p = q = 2$ for the spaces of sequences.

We shall more precisely prove (Bohnenblust-Bunjakovski)-Cauchy-Schwarz-(Sobčyk)'s inequality

$$\sum_{i=1}^{+\infty} |x_i \bar{y}_i| = \sum_{i=1}^{\infty} |x_i| \cdot |y_i| \leq \|x\|_2 \cdot \|y\|_2,$$

if $x, y \in \ell^2$.

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Using that $x + \lambda y \in \ell^2$ for every $\lambda \in \mathbb{R}$, we get

$$\begin{aligned} 0 &\leq \|x + \lambda y\|_2^2 = \sum_{i=1}^{+\infty} (x_i + \lambda y_i) \cdot (\bar{x}_i + \lambda \bar{y}_i) \\ &= \sum_{i=1}^{+\infty} \{ |x_i|^2 + \lambda^2 |y_i|^2 + \lambda \bar{x}_i y_i + \lambda x_i \bar{y}_i \} \\ &= \lambda^2 \sum_{i=1}^{+\infty} |y_i|^2 + \lambda \left\{ \sum_{i=1}^{+\infty} \bar{x}_i y_i + \sum_{i=1}^{+\infty} x_i \bar{y}_i \right\} + \sum_{i=1}^{+\infty} |x_i|^2, \end{aligned}$$

which we write in the form

$$\lambda^2 \cdot \|y\|_2^2 + \lambda \left\{ \sum_{i=1}^{+\infty} \bar{x}_i y_i + \sum_{i=1}^{+\infty} x_i \bar{y}_i \right\} + \|x\|_2^2 \geq 0.$$

This must hold for every real $\lambda \in \mathbb{R}$, so we must have

$$\begin{aligned} 0 &\geq B^2 - 4AC = \left\{ \sum_{i=1}^{\infty} \bar{x}_i y_i + \sum_{i=1}^{+\infty} x_i \bar{y}_i \right\}^2 - 4 \|x\|_2^2 \|y\|_2^2 \\ &= 4 \left(\operatorname{Re} \left\{ \sum_{i=1}^{+\infty} \bar{x}_i y_i \right\} - \{ \|x\|_2 \|y\|_2 \}^2 \right), \end{aligned}$$

hence

$$\left| \operatorname{Re} \left\{ \sum_{i=1}^{\infty} \bar{x}_i y_i \right\} \right| \leq \|x\|_2 \|y\|_2.$$

When x_i and y_i are all real, the inequality follows immediately.

In general,

$$\begin{aligned} \sum_{i=1}^{+\infty} |x_i \bar{y}_i| &= \sum_{i=1}^{+\infty} |x_i| \cdot |\bar{y}_i| \leq \|x\|_2 \cdot \|y\|_2 \\ &= \left\{ \sum_{i=1}^{+\infty} |x_i|^2 \right\}^{\frac{1}{2}} \left\{ \sum_{i=1}^{\infty} |y_i|^2 \right\}^{\frac{1}{2}} = \|x\|_2 \|y\|_2, \end{aligned}$$

and the claim is proved.

Example 2.26 Let $w(t) \geq 0$ be a non-negative function on \mathbb{R} . We define a linear functional I_w by

$$I_w(f) = \int_{\mathbb{R}} f(t) w(t) dt,$$

for $f \in L^1(\mathbb{R})$.

Assume that $|f|^p w$ and $|g|^q w$ are in $L^1(\mathbb{R})$, where f and g are (measurable) functions and $1 < p, q < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$.

1. Show the generalized Hölder's inequality

$$|I_w(fg)| \leq \{I_w(|f|^p)\}^{\frac{1}{p}} \{I_w(|g|^q)\}^{\frac{1}{q}},$$

where the inequality for $w = 1$ can be taken to be valid.

Now recall the Gamma function,

$$\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t} dt, \quad x > 0,$$

with the property $\Gamma(x + 1) = x\Gamma(x)$ for $x > 0$.

2. Use the generalized Hölder's inequality with

$$w(t) = t^{n-1} e^{-t}, \quad 0 < t < \infty, \quad \text{and} \quad p = q = 1,$$

to show that

$$\Gamma\left(n + \frac{1}{2}\right) \leq \frac{n!}{\sqrt{n}}, \quad n \in \mathbb{N}.$$

3. Give a similar estimation of $\Gamma(n + 1)$ by taking

$$w(t) = t^{n-\frac{1}{2}} e^{-t}, \quad 0 < t < \infty, \quad \text{and} \quad p = q = 2,$$

and deduce that

$$\frac{n!}{\sqrt{n + \frac{1}{2}}} \leq \Gamma\left(n + \frac{1}{2}\right) \leq \frac{n!}{\sqrt{n}}, \quad n \in \mathbb{N}.$$

1) We get from $w(t) \geq 0$ that both $w^{1/p}$ and $w^{1/q}$ are defined and that $w^{1/p} \cdot w^{1/q} = w$, and $f \cdot w^{1/p} \in L^p(\mathbb{R})$ and $g \cdot w^{1/q} \in L^q(\mathbb{R})$. Applying the usual Hölder's inequality we get

$$\begin{aligned} |I_w(f \cdot g)| &= \left| \int_{-\infty}^{+\infty} f(t) g(t) w(t) dt \right| \leq \int_{-\infty}^{+\infty} |f(t) w^{\frac{1}{p}}(t)| \cdot |g(t) w^{\frac{1}{q}}(t)| dt \\ &\leq \left\{ \int_{-\infty}^{+\infty} |f(t)|^p w(t) dt \right\}^{\frac{1}{p}} \left\{ \int_{-\infty}^{+\infty} |g(t)|^q w(t) dt \right\}^{\frac{1}{q}} = \{I_w(|f|^p)\}^{\frac{1}{p}} \{I_w(|g|^q)\}^{\frac{1}{q}}, \end{aligned}$$

and we have proved the generalized Hölder's inequality.

2) Then apply this generalized inequality on $f(t) = \sqrt{t} \cdot 1_{\mathbb{R}_+}(t)$ and $g(t) = 1$, and $w(t) = t^{n-1} e^{-t} \cdot 1_{\mathbb{R}_+}(t)$, we get

$$\begin{aligned} \Gamma\left(n + \frac{1}{2}\right) &= \int_0^{+\infty} \sqrt{t} \cdot 1 \cdot t^{n-1} e^{-t} dt \leq \{I_w(t)\}^{\frac{1}{2}} \{I_w(1)\}^{\frac{1}{2}} \\ &= \left\{ \int_0^{+\infty} t \cdot t^{n-1} e^{-t} dt \right\}^{\frac{1}{2}} \left\{ \int_0^{+\infty} 1 \cdot t^{n-1} e^{-t} dt \right\}^{\frac{1}{2}} \\ &= \left\{ \int_0^{+\infty} t^n e^{-t} dt \right\}^{\frac{1}{2}} \left\{ \int_0^{+\infty} t^{n-1} e^{-t} dt \right\}^{\frac{1}{2}} \\ &= \{\Gamma(n+1)\}^{\frac{1}{2}} \{\Gamma(n)\}^{\frac{1}{2}} = \{n!(n-1)!\}^{\frac{1}{2}} = \left\{ \frac{(n!)^2}{n} \right\}^{\frac{1}{2}} = \frac{n!}{\sqrt{n}}. \end{aligned}$$

3) Finitely, let $f(t) = \sqrt{t} \cdot 1_{\mathbb{R}_+}(t)$ and $g(t) = 1$, and $w(t) = t^{n-\frac{1}{2}} e^{-t} \cdot 1_{\mathbb{R}_+}(t)$. Then we get with $p = q = 2$,

$$\begin{aligned} n! &= \Gamma(n+1) = \int_0^{+\infty} t^n e^{-t} dt = \int_0^{+\infty} \sqrt{t} \cdot 1 \cdot t^{n-\frac{1}{2}} e^{-t} dt \\ &\leq \left\{ \int_0^{+\infty} t^{t+\frac{1}{2}} e^{-t} dt \right\}^{\frac{1}{2}} \left\{ \int_0^{+\infty} t^{n-\frac{1}{2}} e^{-t} dt \right\}^{\frac{1}{2}} \\ &= \left\{ \Gamma\left(n + \frac{3}{2}\right) \Gamma\left(n + \frac{1}{2}\right) \right\}^{\frac{1}{2}} = \left\{ \left(n + \frac{1}{2}\right) \left[\Gamma\left(n + \frac{1}{2}\right)\right]^2 \right\}^{\frac{1}{2}} = \sqrt{n + \frac{1}{2}} \cdot \Gamma\left(n + \frac{1}{2}\right), \end{aligned}$$

and we have

$$\frac{n!}{\sqrt{n + \frac{1}{2}}} \leq \Gamma\left(n + \frac{1}{2}\right) \leq \frac{n!}{\sqrt{n}}.$$

Remark 2.2 Furthermore, if we use that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$, it follows from the functional equation that

$$\begin{aligned} \Gamma\left(n + \frac{1}{2}\right) &= \left(n - \frac{1}{2}\right) \Gamma\left(n - \frac{1}{2}\right) = \dots = \left(n - \frac{1}{2}\right) \left(n - \frac{3}{2}\right) \dots \frac{3}{2} \cdot \frac{1}{2} \Gamma\left(\frac{1}{2}\right) \\ &= \frac{(2n-1)(2n-3) \dots 3 \cdot 1}{2 \cdot 2 \dots 2 \cdot 2} \sqrt{\pi} \\ &= \frac{\sqrt{\pi}}{2^n} \cdot \frac{2n}{2n} \cdot \frac{2n-1}{1} \cdot \frac{2n-2}{2(n-1)} \dots \frac{4}{2 \cdot 2} \cdot \frac{3}{1} \cdot \frac{2}{2 \cdot 1} \cdot \frac{1}{1} \\ &= \frac{\sqrt{\pi}}{2^n} \cdot \frac{(2n)!}{2^n \cdot n!} = \frac{\sqrt{\pi}}{4^n} \binom{2n}{n} n!, \end{aligned}$$

hence by insertion

$$\frac{n!}{\sqrt{n + \frac{1}{2}}} \leq \frac{\sqrt{\pi}}{4^n} \binom{2n}{n} n! \leq \frac{n!}{\sqrt{n}},$$

thus

$$\frac{4^n}{\sqrt{\pi \left(n + \frac{1}{2}\right)}} \leq \binom{2n}{n} \leq \frac{4^n}{\sqrt{\pi n}},$$


which is in agreement with Stirling's formula

$$n! \sim \sqrt{2\pi} \cdot n^{n+\frac{1}{2}} e^{-n},$$

because

$$\binom{2n}{n} = \frac{(2n)!}{(n!)^2} \sim \frac{\sqrt{2\pi} \cdot (2n)^{2n+\frac{1}{2}} e^{-2n}}{\left\{\sqrt{2\pi} \cdot n^{n+\frac{1}{2}} \cdot e^{-n}\right\}^2} = \frac{1}{\sqrt{2\pi}} \cdot \frac{(2n)^{2n+\frac{1}{2}}}{n^{2n+1}} = \frac{2^{2n}\sqrt{2}}{\sqrt{2\pi}n} = \frac{4^n}{\sqrt{\pi n}}. \quad \diamond$$

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Example 2.27 *Let*

$$F = \{f \in C^2([0, 1]) \mid f(0) = f(1) = 0\} \subseteq L^2([0, 1]).$$

1) Show that $\|f'\|_2^2 \leq \|f\|_2 \cdot \|f''\|_2$ for $f \in F$.

2) Let $f \in F$. Show that $|f(x)| \leq \|f'\|_2 \sqrt{x}$ for $0 \leq x \leq 1$, and deduce that

$$\|f\|_2 \leq \frac{1}{\sqrt{2}} \|f'\|_2.$$

3) Show that for $f \in C^2([0, 1])$ with $f(0) = f(1) = 0$ we have

$$\|f'\|_2 \leq \frac{1}{\sqrt{2}} \|f''\|_2.$$

4) Show by a counterexample that the result from question (3) is not valid for general $f \in C^2([0, 1])$.

1) We deduce from $f \in C^2([0, 1])$ and $f(0) = f(1) = 0$ and a partial integration, followed by an application of the Cauchy-Schwarz inequality that

$$\begin{aligned} \|f'\|_2^2 &= \int_0^1 f'(t) \overline{f'(t)} dt = \left[f(t) \overline{f'(t)} \right]_0^1 - \int_0^1 f(t) \overline{f''(t)} dt \\ &\leq 0 + \int_0^1 |f(t)| \cdot |f''(t)| dt \leq \|f\|_2 \cdot \|f''\|_2. \end{aligned}$$

2) From

$$f(x) = f(0) + \int_0^x f'(t) dt = \int_0^1 1_{[0,x]}(t) f'(t) dt$$

follows by Cauchy-Schwarz's inequality that

$$|f(x)| = \left| \int_0^1 1_{[0,x]}(t) f'(t) dt \right| \leq \|1_{[0,x]}\|_2 \cdot \|f'\|_2 = \sqrt{x} \cdot \|f'\|_2,$$

where we have used that

$$\|1_{[0,x]}\|_2 = \sqrt{\int_0^1 1_{[0,x]}(t) dt} = \sqrt{\int_0^x 1 dt} = \sqrt{x}.$$

3) Let $f \in F$. It follows from (1) and (2) that

$$\begin{aligned} \|f'\|_2^2 &\leq \|f\|_2 \cdot \|f''\|_2 = \left\{ \int_0^1 |f(x)|^2 dx \right\}^{\frac{1}{2}} \cdot \|f''\|_2 \leq \left\{ \int_0^1 x \|f'\|_2^2 dx \right\}^{\frac{1}{2}} \|f''\|_2 \\ &= \left\{ \int_0^1 x dx \right\}^{\frac{1}{2}} \|f'\|_2 \cdot \|f''\|_2 = \frac{1}{\sqrt{2}} \|f'\|_2 \|f''\|_2. \end{aligned}$$

If $\|f'\|_2 = 0$, the inequality is obvious.

If $\|f'\|_2 > 0$, we obtain the inequality when we divide by $\|f'\|_2$.

We derived the above by assuming that $f \in F$, thus $f(0) = f(1) = 1$.

Now, let $f(0) = f(1) = c$. Then $f(x) - c \in F$, hence

$$\|f'\|_2 = \|(f - c)'\|_2 \leq \frac{1}{\sqrt{2}} \|(f - c)''\|_2 = \frac{1}{\sqrt{2}} \|f''\|_2.$$

4) Finally, let $f(x) = ax$. Then $f'(x) = a$ and $f''(x) = 0$, hence

$$\|f'\|_2 = |a| \quad \text{og} \quad \|f''\|_2 = 0,$$

and the inequality is not fulfilled for any $a \neq 0$.

Example 2.28 1) Let $1 \leq p \leq q \leq \infty$. Show that $\ell^p \subset \ell^q$.

2) Let $1 \leq r < p < 2r$ and assume that the sequence (x_n) satisfies

$$\sum_{n=1}^{\infty} n |x_n|^p < \infty.$$

Show that $(x_n) \in \ell^r$.

1) If $(x_n) \in \ell^p$, then $\sum_{n=1}^{+\infty} |x_n|^p < +\infty$. In particular, $x_n \rightarrow 0$ for $n \rightarrow +\infty$, hence there exists an $N \in \mathbb{N}$, such that $|x_n| < 1$ for all $n \geq N + 1$.

For $p = q$ there is nothing to prove. If $1 \leq p < q < +\infty$, then

$$\sum_{n=1}^{+\infty} |x_n|^q = \sum_{n=1}^N |x_n|^q + \sum_{n=N+1}^{+\infty} |x_n|^p \cdot |x_n|^{q-p} < \sum_{n=1}^N |x_n|^q + \sum_{n=N+1}^{+\infty} |x_n|^p < +\infty,$$

showing that $(x_n) \in \ell^q$.

If $1 \leq p < q = +\infty$, then clearly

$$\sup_{n \in \mathbb{N}} |x_n| \leq \max \{1, \sup\{|x_n| \mid n = 1, \dots, N\}\} < +\infty,$$

and we conclude that $(x_n) \in \ell^\infty$.

2) Then let $1 \leq r < p < 2r$ and assume that

$$\sum_{n=1}^{+\infty} n |x_n|^p < +\infty.$$

Let $0 < s < 1$. We shall somehow way apply Hölder's inequality with $\tilde{p} = \frac{1}{s} > 1$ and $\tilde{q} = \frac{1}{1-s} > 1$. The assumption shall also be applied later on, so we get by a reasonable rewriting and an application of Hölder's inequality,

$$\sum_{n=1}^{+\infty} |x_n|^r = \sum_{n=1}^{+\infty} \{n |x_n|^p\}^s \left\{ \frac{1}{n^s} |x_n|^{r-sp} \right\} \leq \left\{ \sum_{n=1}^{+\infty} n |x_n|^p \right\}^s \cdot \left\{ \sum_{n=1}^{+\infty} n^{-\frac{s}{1-s}} |x_n|^{\frac{r-sp}{1-s}} \right\}^{1-s}.$$

By the assumption, the former factor is finite for every $s \in]0, 1[$. The task is to choose s in this interval, such that the latter factor also becomes finite.

Using that $2r > p$, we get $\frac{r-sp}{1-s} = 0$ for $s = \frac{r}{p} > \frac{1}{2}$. We get with this s that $\alpha = \frac{s}{1-s} > 1$ and

$$\sum_{n=1}^{+\infty} n^{-\frac{s}{1-s}} |x_n|^{\frac{r-sp}{1-s}} = \sum_{n=1}^{+\infty} \frac{1}{n^\alpha} \cdot |x_n|^0 = \sum_{n=1}^{+\infty} \frac{1}{n^\alpha} < +\infty,$$

and the latter factor in the estimate above is finite for this particular $s = \frac{r}{p} \in \left] \frac{1}{2}, 1 \right[$. Now, s does not occur in the sum, we are estimating, so we conclude that

$$\sum_{n=1}^{+\infty} |x_n|^r < +\infty,$$

and we have proved that $(x_n) \in \ell^r$.

Example 2.29 Define in \mathbb{R}^2 the function

$$\|x\| = \|(x_1, x_2)\| = \left(\sqrt{|x_1|} + \sqrt{|x_2|} \right)^2.$$

Is it a norm?

Sketch the set $\{(x_1, x_2) \mid \|(x_1, x_2)\| \leq 1\}$.

First note that $\|x\| = \|x\|_p$, where $p = \frac{1}{2} < 1$.

The first two conditions of a norm are trivially fulfilled, so we shall only consider the triangle inequality. We shall prove that it is *not* satisfied. It suffices to find two vectors x and y , for which the triangle inequality does not hold.

Choose $x = (1, 0)$ and $y = (0, 1)$. Then $\|x\| = \|y\| = 1$, and

$$\|x + y\| = \|(1, 1)\| = (\sqrt{1} + \sqrt{1})^2 = 4,$$

hence

$$\|x + y\| = 4 > 2 = \|x\| + \|y\|,$$

and the triangle inequality is not fulfilled, and $\|\cdot\|$ is not a norm.

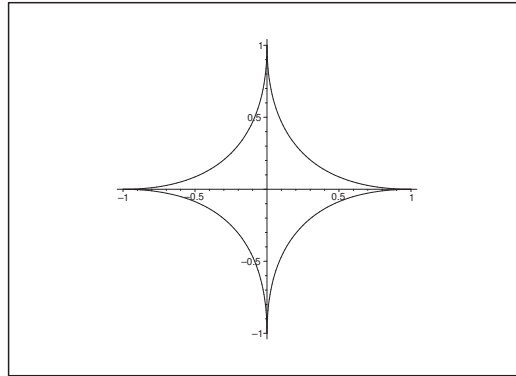


Figure 5: The unit “ball” corresponding to $\|\cdot\|$.

Remark 2.3 It is not hard to prove that *if* $\|\cdot\|$ is a norm, then the corresponding unit ball is convex. (However, not every convex set will induce a norm).

Since the set, which should be the unit ball clearly is *not* convex (cf. the figure), $\|\cdot\|$ is *not* a norm. \diamond

Remark 2.4 Even if $\|\cdot\|_{\frac{1}{2}}$ is not a norm in the usual sense, there exist some applications of it, e.g. in the theory of H^p spaces in Complex Function Theory, and the “norm” of such functions can nevertheless be given a reasonable interpretation. \diamond

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3 Bounded operators

Example 3.1 Let T be a linear operator from a normed space V into a normed space W .

Show that the image $T(V)$ is a subspace of W .

Show that the kernel (or null-space) $\ker(T)$ is a subspace of V .

If T is bounded, is it true that $T(V)$ and/or $\ker(T)$ are closed?

1) Let $w_1, w_2 \in T(V) \subseteq W$, and let λ be a scalar. We shall prove that

$$w_1 + \lambda w_2 \in T(V).$$

Remark 3.1 It is here of paramount importance that the field of the scalars is the same both places. If e.g. $T : V \rightarrow W$ is given by

$$Tx = x + i \cdot 0,$$

where $V = (\mathbb{R}, +, \cdot, \|\cdot\|, \mathbb{R})$ and $W = (\mathbb{C}, +, \cdot, \|\cdot\|, \mathbb{C})$, then T is linear, and $T(V)$ is a subspace of the 2-dimensional space $(\mathbb{C}, +, \cdot, \|\cdot\|, \mathbb{R})$ over \mathbb{R} . It is, however, not a subspace of the 1-dimensional space $W = (\mathbb{C}, +, \cdot, \|\cdot\|, \mathbb{C})$ over \mathbb{C} , so the claim is not true in this case. \diamond

It follows from the assumption $w_1, w_2 \in T(V)$ that there exist v_1 and $v_2 \in V$, such that $w_1 = Tv_1$ and $w_2 = Tv_2$.

If we put $v = v_1 + \lambda v_2 \in V$, then

$$T(V) \ni Tv = T(v_1 + \lambda v_2) = Tv_1 + \lambda Tv_2 = w_1 + \lambda w_2.$$

2) Now $\ker(T) = \{v \in V \mid Tv = 0\}$, and T is linear. Hence, if $v_1, v_2 \in \ker(T)$, and λ is a scalar, then

$$T(v_1 + \lambda v_2) = Tv_1 + \lambda Tv_2 = 0 + \lambda \cdot 0 = 0,$$

thus $v_1 + \lambda v_2 \in \ker(T)$, and $\ker(T)$ is a subspace.

3) If T is bounded, then T is continuous. Now $\{0\} \subset W$ is closed, so $\ker(T) = T^{-1}(\{0\})$ is closed.

On the other hand, $T(V)$ need not be closed, which is demonstrated by the example below.

Choose $V = W = C^0([0, 1])$ with the norm $\|\cdot\|_\infty$, and let $T : V \rightarrow W$ be given by

$$Tf(t) = \int_0^t f(s) dx, \quad t \in [0, 1].$$

Then T is bounded,

$$|Tf(t)| = \left| \int_0^t f(s) ds \right| \leq \int_0^t |f(s)| ds \leq \int_0^1 |f(s)| ds \leq 1 \cdot \|f\|_{\text{inf ty}}, \quad t \in [0, 1],$$

hence

$$\|Tf\|_\infty \leq 1 \cdot \|f\|_\infty, \quad \|T\| \leq 1.$$

Furthermore,

$$T(V) = \{w \in C^1([0, 1]) \mid w(0) = 0\}$$

is dense in

$$\{w \in C^0([0, 1]) \mid w(0) = 0\} \subset W,$$

without being equal to it.

That $T(V)$ is dense, is seen in the following way: Every polynomial of constant term 0 lies in $T(V)$. The claim then follows by a suitable variant of Weierstraß's Approximation Theorem.

There exist of course C^0 -functions which are not of class C^1 , hence $T(V)$ is not equal to the smallest closed subspace

$$\{w \in C^0([0, 1]) \mid w(0) = 0\}$$

which contains $T(V)$ (because $T(V)$ is dense in this space).

Example 3.2 In the Banach space ℓ^p , $1 \leq p \leq \infty$, we have a sequence (x_n) converging to an element x , where

$$x_n = (x_{n1}, x_{n2}, \dots) \quad \text{and} \quad x = (x_1, x_2, \dots).$$

Show that if $x_n \rightarrow x$ in ℓ^p , then $x_{nk} \rightarrow x_k$ for all $k \in \mathbb{N}$.

If $x_{nk} \rightarrow x_k$ for all $k \in \mathbb{N}$, is it true that $x_n \rightarrow x$ in ℓ^p ?

Let $x_n \rightarrow x$ in ℓ^p , $1 \leq p < \infty$, thus $\|x - x_n\|_p \rightarrow 0$ for $n \rightarrow \infty$, i.e.

$$\sum_{k=1}^{\infty} |x_k - x_{nk}|^p = \|x - x_n\|_p^p \rightarrow 0 \quad \text{for } n \rightarrow \infty.$$

If $p = \infty$, then $x_n \rightarrow x$ in ℓ^∞ means that

$$\|x - x_n\|_\infty = \sup_k |x_k - x_{nk}| \rightarrow 0 \quad \text{for } n \rightarrow \infty.$$

In both cases we get for every fixed k that

$$|x_k - x_{nk}| \leq \|x - x_n\|_p \rightarrow 0 \quad \text{for } n \rightarrow \infty,$$

thus $x_{nk} \rightarrow x_k$ for $n \rightarrow \infty$, and the first claim is proved.

On the other hand, if $x_{nk} \rightarrow x_k$ for every fixed k , then we cannot conclude that $x_n \rightarrow x$ in ℓ^p . Just choose

$$x_n = (\delta_{nk}) = (0, \dots, 0, 1, 0, \dots)$$

with 1 on place number n , and 0 otherwise.

We have for this sequence that $x_{nk} \rightarrow 0$ for every fixed k , thus $x = 0$.

On the other hand,

$$\|x_n\|_p = \|x_n - 0\|_p = \left\{ \sum_{k=1}^{\infty} |\delta_{nk}|^p \right\}^{\frac{1}{p}} = 1 \quad \text{for } 1 \leq p < +\infty,$$

and

$$\|x_n\|_{\infty} = \|x_n - 0\|_{\infty} = 1,$$

so none of these sequences converges towards, i.e. the sequence does not converge in any ℓ^p , $1 \leq p \leq +\infty$.

Example 3.3 Let T be a linear mapping from \mathbb{R}^m to \mathbb{R}^n , both equipped with the 2-norm. Let (a_{ij}) denote a real $n \times m$ matrix corresponding to T . Show that T is a bounded linear operator with $\|T\|^2 \leq \sum_i \sum_j a_{ij}^2$.

We get (cf. EXAMPLE 1.23)

$$\begin{aligned} \|Tx\|_2^2 &= \left\| \left(\sum_{j=1}^m a_{ij}x_j \right)_{i \in \mathbb{N}} \right\|_2^2 = \sum_{i=1}^n \left\{ \sum_{j=1}^m a_{ij}x_j \right\}^2 = \sum_{i=1}^n \sum_{j=1}^m \sum_{k=1}^m a_{ij}x_j a_{ik}x_k \\ &= \sum_{i=1}^n \sum_{j=1}^m \sum_{k=1}^m (a_{ij}x_k) \cdot (a_{ik}x_j). \end{aligned}$$

Then note that

$$|a_{ij}x_k| \cdot |a_{ik}x_j| \leq \frac{1}{2} a_{ij}^2 x_k^2 + \frac{1}{2} a_{ik}^2 x_j^2.$$

By insertion of this inequality,

$$\begin{aligned} \|Tx\|_2^2 &= \sum_{i=1}^n \sum_{j=1}^m \sum_{k=1}^m (a_{ij}x_k) \cdot (a_{ik}x_j) \leq \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^m \sum_{k=1}^m a_{ij}^2 x_k^2 + \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^m \sum_{k=1}^m a_{ik}^2 x_j^2 \\ &= 2 \cdot \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^m a_{ij}^2 \cdot \sum_{k=1}^m x_k^2 = \sum_{i=1}^n \sum_{j=1}^m a_{ij}^2 \cdot \|z\|_1^2. \end{aligned}$$

Since $\|T\|^2$ is the smallest constant, for which we have such an estimate, we have

$$\|T\|^2 \leq \sum_{i=1}^n \sum_{j=1}^m a_{ij}^2.$$

Example 3.4 Let T be a linear operator from a normed space V into a normed space W , and assume that V is finite dimensional. Show that T must be bounded.

The space V is finite dimensional, thus we can choose a basis e_1, \dots, e_n for V , where $\|e_k\|_V = 1$. Then for every $v \in V$,

$$\begin{aligned} \|Tv\|_W &= \left\| T \left(\sum_{j=1}^n \lambda_j e_j \right) \right\|_W = \left\| \sum_{j=1}^n \lambda_j T e_j \right\|_W \leq \sum_{j=1}^n |\lambda_j| \cdot \|T e_j\|_W \\ &\leq \max \{ \|T e_j\|_W \mid j = 1, \dots, n \} \cdot \sum_{j=1}^n |\lambda_j|. \end{aligned}$$

If we can prove that there exists a constant $c > 0$, such that

$$(12) \quad \sum_{j=1}^n |\lambda_j| \leq c \left\| \sum_{j=1}^n \lambda_j e_j \right\|_V \quad \text{for every } \lambda_1, \dots, \lambda_n,$$

then

$$\|Tv\|_W \leq c \cdot \max_j \|T e_j\|_W \cdot \|v\|_V,$$

which shows that T is bounded

$$\|T\| \leq c \cdot \max_{j=1, \dots, n} \|T e_j\|_W.$$

We shall therefore only prove (12).

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INDIRECT PROOF. Assume that (12) does *not* hold, i.e. assume that

$$(13) \quad \forall N \in \mathbb{N} \exists \lambda_{N,1}, \dots, \lambda_{N,n} : \sum_{j=1}^n |\lambda_{N,j}| > N \left\| \sum_{j=1}^n \lambda_{N,j} e_j \right\|_V.$$

Due to the homogeneity we may assume that $\lambda_{N,j}$ is chosen, such that

$$\sum_{j=1}^n |\lambda_{N,j}| = 1.$$

Then it follows from (13) that $\|v_N\|_V \leq \frac{1}{N}$, hence

$$v_N = \sum_{j=1}^n \lambda_{N,j} e_j \rightarrow 0 \quad \text{for } N \rightarrow \infty.$$

Now, e_1, \dots, e_n is a basis for V , hence $\lambda_{N,j} \rightarrow 0$ for $N \rightarrow \infty$ for every $j = 1, \dots, n$. In particular, there is an $N_0 \in \mathbb{N}$, such that for every $N \geq N_0$ we have $|\lambda_{N,j}| < \frac{1}{2n}$. This gives us the following *contradiction*

$$1 = \sum_{j=1}^n |\lambda_{N,j}| < \sum_{j=1}^n \frac{1}{2n} = \frac{1}{2}.$$

We have now proved that (13) does *not* hold, hence (12) holds instead, and as proved previously (12) implies that T is bounded, and the claim is proved.

Example 3.5 Let T be a linear operator from a finite dimensional vector space into itself. Show that T is injective if and only if T is surjective.

Let $T : V \rightarrow V$ be linear, where $\dim V = n$. Let e_1, \dots, e_n form a basis. Now, T is linear, so T is injective, if and only if $Tu = Tv$, i.e. $T(u-v) = 0$ implies that $u = v$, or put in another way, $u-v = 0$. Thus T is injective, if and only if

$$(14) \quad Tv = 0 \implies v = 0.$$

Now assume that T is injective. We shall prove that $Te_1, \dots, Te_n \in V$ are linearly independent.

Assume that $\lambda_1 Te_1 + \dots + \lambda_n Te_n = 0$. Then by the linearity,

$$0 = \lambda_1 Te_1 + \dots + \lambda_n Te_n = T(\lambda_1 e_1 + \dots + \lambda_n e_n),$$

and we conclude using (14) that

$$\lambda_1 e_1 + \dots + \lambda_n e_n = 0.$$

Since e_1, \dots, e_n is a basis for V , we must have $\lambda_1 = \dots = \lambda_n = 0$, and it follows that Te_1, \dots, Te_n are n linearly independent vectors in the image $T(V)$. Then

$$n \geq \dim T(V) \geq n, \quad \text{thus} \quad \dim T(V) = n,$$

hence $T(V) = V$, and we have proved that T is surjective.

Assume conversely that T is surjective. To the basis formed by $e_1, \dots, e_n \in V$ corresponds the vectors $f_1, \dots, f_n \in V$, where

$$Tf_1 = e_1, \quad \dots, \quad Tf_n = e_n.$$

If $\lambda_1 f_1 + \dots + \lambda_n f_n = 0$, then we conclude that

$$0 = T(\lambda_1 f_1 + \dots + \lambda_n f_n) = \lambda_1 Tf_1 + \dots + \lambda_n Tf_n = \lambda_1 e_1 + \dots + \lambda_n e_n.$$

Using again that e_1, \dots, e_n form a basis for V , we infer that $\lambda_1 = \dots = \lambda_n = 0$, which again implies that f_1, \dots, f_n form a basis for V .

If $v = \lambda_1 f_1 + \dots + \lambda_n f_n$ satisfies $Tv = 0$, then

$$0 = Tv = T(\lambda_1 f_1 + \dots + \lambda_n f_n) = \lambda_1 Tf_1 + \dots + \lambda_n Tf_n = \lambda_1 e_1 + \dots + \lambda_n e_n,$$

and we infer again that $\lambda_1 = \dots = \lambda_n = 0$, hence $v = 0$, and (14) is fulfilled, so T is injective.

Example 3.6 Let T be the linear mapping from $C^\infty(\mathbb{R})$ into itself given by $Tf = f'$. Show that T is surjective? Is T injective?

Let $f \in C^\infty(\mathbb{R})$. Define $g \in C^\infty(\mathbb{R})$ by

$$g(t) = \int_0^t f(s) ds, \quad t \in \mathbb{R}.$$

Clearly, $Tg = g$, so $T(V) = C^\infty(\mathbb{R})$, and T is surjective.

Define instead

$$g_1(t) = 1 + \int_0^t f(s) ds = 1 + g(t) \in C^\infty(\mathbb{R}).$$

Then

$$Tg_1 = f = Tg,$$

and since $g_1 \neq g$, it follows that T is not injective.

Example 3.7 Let $I = [a, b]$ be a bounded interval and consider the linear mapping T from $C([a, b])$ into itself, given by

$$Tf(t) = \int_a^t f(s) ds.$$

We assume that $C([a, b])$ is equipped with the sup-norm.

Show that T is bounded and find $\|T\|$.

Show that T is injective and find $T^{-1} : T(C([a, b])) \rightarrow C([a, b])$.

Is T^{-1} bounded?

When

$$Tf(t) = \int_a^t f(s) ds \quad \text{for } t \in [a, b],$$

then

$$|Tf(t)| = \left| \int_a^t f(s) ds \right| \leq \int_a^t |f(s)| ds \leq \|f\|_\infty \int_a^t ds = (t-a)\|f\|_\infty \leq (b-a) \cdot \|f\|_\infty,$$

thus

$$\|Tf\|_\infty \leq (b-a) \cdot \|f\|_\infty,$$

proving that T is bounded and $\|T\| \leq b-a$.

Choose $f(t) = 1$ for every $t \in [a, b]$. Then $\|f\|_\infty = 1$, and

$$Tf(t) = \int_a^t ds = t-a \quad \text{for } t \in [a, b],$$

hence

$$\|Tf\|_\infty = \sup_{t \in [a, b]} (t-a) = b-a,$$

and we conclude that $\|T\| \geq b-a$, whence by the previously proved result, $\|T\| = b-a$.

Assume that

$$Tf(t) = \int_a^t f(s) ds \equiv 0.$$

Since $f \in C([a, b])$, we have $Tf \in C^1([a, b])$ with

$$\frac{d}{dt} Tf(t) = f(t) \equiv 0,$$

which shows that $f \equiv 0$, so T is injective.

It follows from the above that $T(C([a, b])) \subseteq C^1([a, b])$. We get from $Tf(a) = 0$ that even

$$T(C([a, b])) \subseteq \{g \in C^1([a, b]) \mid g(a) = 0\}.$$

Conversely, if $g \in C^1([a, b])$ and $g(a) = 0$, then $f = g' \in C([a, b])$, and $Tf = g$, and the image becomes

$$T(C([a, b])) = \{g \in C^1([a, b]) \mid g(a) = 0\}.$$

Finally, it is immediately seen that

$$T^{-1} : T(C([a, b])) \rightarrow C([a, b])$$

is given by $T^{-1}g = g'$.

The operator T^{-1} is not bounded. We have e.g. that $(t - a)^n \in T(C([a, b]))$, and

$$\|(t - a)^n\|_\infty = \sup_{t \in [a, b]} |(t - a)^n| = (b - a)^n.$$

It follows from $T^{-1}(t - a)^n = n(t - a)^{n-1}$ that

$$\|T^{-1}(t - a)^n\|_\infty = n(b - a)^{n-1} = \frac{n}{b - a} \|(t - a)^n\|_\infty,$$

proving that there is no constant $c > 0$, such that

$$\|T^{-1}f\|_\infty \leq c \|f\|_\infty, \quad \text{for all } f \in T(C([a, b])),$$

and T is not bounded.

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Example 3.8 Let T be a bounded linear operator from a normed vector space V into a normed vector space W , and assume that T is surjective. Assume that there is a $c > 0$, such that

$$\|Tx\| \geq c\|x\| \quad \text{for all } x \in V.$$

show that T^{-1} exists and that $T^{-1} \in B(W, V)$.

We require that T^{-1} exists, so we shall first prove that T is injective, i.e. if $Tx = Ty$, then $x = y$.

The mapping T is linear, so this is equivalent with that $T(x - y) = 0$ implies that $x - y = 0$, or by a slight change of notation:

Assume that $Tx = 0$. Prove that $x = 0$.

When $Tx = 0$, then it follows from the assumption that

$$0 \leq \|x\| \leq \frac{1}{c} \|Tx\| = 0, \quad \text{thus } \|x\| = 0, \text{ hence } x = 0,$$

and the claim is proved.

We have proved that T is injective, thus T^{-1} exists. Now $T(V) = W$, so $T^{-1} : W \rightarrow V$, and T^{-1} is defined on all of W . It remains only to be proved that T^{-1} is bounded.

Let $y \in W$. Then $x = T^{-1}y$ is defined. It follows from the assumption that

$$\|T^{-1}y\| = \|x\| \leq \frac{1}{c} \|Tx\| = \frac{1}{c} \|T(T^{-1}y)\| = \frac{1}{c} \|y\|,$$

which shows that T^{-1} is bounded, $\|T^{-1}\| \leq \frac{1}{c}$, and it follows that $T^{-1} \in B(W, V)$.

Example 3.9 Let V and W be two normed spaces. Prove that $B(V, W)$ is a normed vector space and that $B(V, W)$ is a Banach space, if W is a Banach space.

It is well-known that $B(V, W)$ is a vector space.

Define $\|T\|$ by

$$\|T\| = \sup\{\|Tx\|_W \mid \|x\|_V \leq 1\}.$$

Then clearly, $\|T\| \geq 0$. If $T \neq 0$, then there exists an $x \in V$, such that $Tx \neq 0$, and we conclude that $\|T\| = 0$, if and only if $T = 0$.

Furthermore,

$$\|\alpha T\| = \sup\{\|\alpha Tx\|_W \mid \|x\|_V \leq 1\} = |\alpha| \cdot \sup\{\|Tx\|_W \mid \|x\|_V \leq 1\} = |\alpha| \cdot \|T\|.$$

Finally,

$$\begin{aligned} \|T_1 + T_2\| &= \sup\{\|(T_1 + T_2)x\|_W \mid \|x\|_V \leq 1\} \leq \sup\{\|T_1x\|_W + \|T_2x\|_W \mid \|x\|_V \leq 1\} \\ &\leq \sup\{\|T_1x\|_W \mid \|x\|_V \leq 1\} + \sup\{\|T_2x\|_W \mid \|x\|_V \leq 1\} = \|T_1\| + \|T_2\|, \end{aligned}$$

and we have proved that $\|\cdot\|$ is a norm on $B(V, W)$, and $B(V, W)$ is a normed vector space.

We now assume that W is a Banach space, thus every Cauchy sequence on W is convergent. We shall prove that $B(V, W)$ becomes a Banach space with the norm introduced above. Let (T_n) be a Cauchy sequence on $B(V, W)$, i.e.

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} \forall m, n \geq N : \|T_m - T_n\| < \varepsilon.$$

Then it follows from the definition that

$$\|T_m - T_n\| = \sup\{\|(T_m - T_n)x\|_W \mid \|x\|_V \leq 1\} = \sup\{\|T_m - T_n\|_W \mid \|x\|_V \leq 1\} < \varepsilon.$$

In particular, we have for every $x \in V$, for which $\|x\|_V \leq 1$ that

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} \forall m, n \geq N : \|T_mx - T_nx\|_W < \varepsilon,$$

which is the condition for (T_nx) being a Cauchy sequence on W . We assumed that W was a Banach space, so it is complete. This implies that (T_nx) is convergent, and it follows that $(T_n(\lambda x)) = (\lambda T_nx)$ is also convergent in W for every λ , and the condition $\|x\|_V \leq 1$ has become superfluous.

Define an operator $T : V \rightarrow W$ by

$$Tx = \lim_{n \rightarrow +\infty} T_nx, \quad x \in V.$$

Then

$$T(x + \lambda y) = \lim_{n \rightarrow +\infty} T_n(x + \lambda y) = \lim_{n \rightarrow +\infty} \{T_nx + \lambda T_ny\} = \lim_{n \rightarrow +\infty} T_nx + \lambda \lim_{n \rightarrow +\infty} T_ny = Tx + \lambda Ty,$$

which shows that T is linear.

It remains to be proved that $T \in B(V, W)$, i.e. that T is bounded. If $x \in V$ with $\|x\|_V \leq 1$, then

$$\|Tx\| = \left\| \lim_{n \rightarrow +\infty} T_nx \right\| \leq \sup_{n \in \mathbb{N}} \|T_nx\| \leq \sup_{n \in \mathbb{N}} \|T_n\|.$$

Since (T_n) is a Cauchy sequence, we have $\sup_{n \in \mathbb{N}} \|T_n\| < +\infty$, and we conclude that $T \in B(V, W)$. Thus we have proved that the Cauchy sequence $(T_n) \subseteq B(V, W)$ converges towards $T \in B(V, W)$, and we have proved that $B(V, W)$ is a Banach space.

Example 3.10 Let $S, T \in B(V, V)$. Prove that the composite mapping ST (defined by $(ST)x = S(Tx)$ for $x \in V$) belongs to $B(V, V)$, and that

$$\|ST\| \leq \|S\| \cdot \|T\|.$$

When $S, T \in B(V, V)$, the composition ST is defined (and linear) on all of V . We shall only prove that ST is bounded. Now, for every $x \in V$,

$$\|(ST)x\|_V = \|S(Tx)\|_V \leq \|S\| \cdot \|Tx\|_V \leq \|S\| \cdot \|T\| \cdot \|x\|_V,$$

so

$$\|ST\| = \sup\{\|(ST)x\|_V \mid \|x\|_V \leq 1\} \leq \sup\{\|S\| \cdot \|T\| \cdot \|x\|_V \mid \|x\|_V \leq 1\} = \|S\| \cdot \|T\|.$$

Example 3.11 Let V be a Banach space and let $T \in B(V)$ be such that T^{-1} exists and belongs to $B(V)$.

Show that if $\|T\|$ and $\|T^{-1}\| \leq 1$, then

$$\|T\| = \|T^{-1}\| = 1,$$

and $\|Tf\| = \|f\|$ for all $f \in V$.

It follows from the assumptions that T is bijective,

$$(15) \quad Tf = g, \quad T^{-1}g = f.$$

We first prove that

$$\|Tf\| = \|f\| \quad \text{for every } f \in V.$$

This follows from

$$\|Tf\| \leq \|T\| \cdot \|f\| = \|f\| = \|T^{-1}f\| \leq \|T^{-1}\| \cdot \|g\| = \|g\| = \|Tf\|.$$

Hence we must have equality everywhere, and in particular,

$$\|Tf\| = \|f\| \quad \text{for all } f \in V,$$

and

$$\|T^{-1}g\| = \|g\| \quad \text{for all } g \in V.$$

Finally, we get

$$\|T\| = \sum\{\|Tf\| \mid \|f\| = 1\} = \sup\{\|f\| \mid \|f\| = 1\} = 1,$$

and

$$\|T^{-1}\| = \sup\{\|T^{-1}g\| \mid \|g\| = 1\} = \sup\{\|g\| \mid \|g\| = 1\} = 1.$$

Example 3.12 Let H denote a Hilbert space and let $T \in B(H)$ and assume that there is a positive c such that

$$|(Tx, x)| \geq c \|x\|^2 \quad \text{for all } x \in H.$$

Show that T^{-1} exists and belongs to $B(H)$.

Assume that $Tx = 0$. Then

$$0 = |(Tx, x)| \geq c \|x\|^2 \geq 0,$$

from which we conclude that $x = 0$, and we have proved that T is injective, so T^{-1} exists.

If $x = T^{-1}y$ for some $y \in H$, then it follows from the estimate

$$c \|x\|^2 = c \|T^{-1}y\|^2 \leq |(y, T^{-1}y)| \leq \|y\| \cdot \|T^{-1}y\|,$$

that $\|T^{-1}\| \leq \frac{1}{c}$, so T^{-1} is bounded on the image $T(H)$.

It remains to prove that the image $T(H)$ is all of H . Let $x \perp T(H)$. Then we get again that

$$0 = |(Tx, x)| \geq c \|x\|^2,$$

which proves that $x = 0$ is the only vector, which is perpendicular to the image, so $\overline{T(H)} = H$. Since T^{-1} is bounded, it has a continuous extension to $\overline{T(H)} = H$, and it follows that $T^{-1} \in B(H)$.

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Example 3.13 Let $p > 1$ and let $f(x, t) \geq 0$ be a (measurable) function on \mathbb{R}^2 such that

$$g(t) = \left\{ \int_{\mathbb{R}} f(x, t) dx \right\}^{p-1}$$

exists.

1) Put $q = \frac{p}{p-1}$ and show that

$$\left\| \int_{\mathbb{R}} f(x, \cdot) dx \right\|_p^p \leq \|g\|_q \int_{\mathbb{R}} \|f(x, \cdot)\|_p dx.$$

2) Let $f(x, t)$ be a (measurable) function on \mathbb{R}^2 such that the function

$$x \mapsto \|f(x, \cdot)\|_p$$

belongs to $L^1(\mathbb{R})$. Use question 1 to show the inequality

$$\left\| \int_{\mathbb{R}} f(x, \cdot) dx \right\|_p \leq \int_{\mathbb{R}} \|f(x, \cdot)\|_p dx,$$

first for $p > 1$, and then for $p = 1$.

3) Let $g \in L^p(\mathbb{R})$ and $h \in L^1(\mathbb{R})$. We define the convolution $g \star h$ by

$$g \star h(t) = \int_{\mathbb{R}} g(t-x) h(x) dx.$$

Show that convolution with an $L^1(\mathbb{R})$ -function is a linear and bounded mapping from $L^p(\mathbb{R})$ into $L^p(\mathbb{R})$ for any $p > 1$.

1) We get

$$\begin{aligned} \left\| \int_{\mathbb{R}} f(x, \cdot) dx \right\|_p^p &= \int_{\mathbb{R}} \left\{ \int_{\mathbb{R}} f(x, t) dx \right\}^p dt = \int_{\mathbb{R}} g(t) \left\{ \int_{\mathbb{R}} f(x, t) dx \right\} dt \\ &= \int_{\mathbb{R}} \left\{ \int_{\mathbb{R}} g(t) \cdot f(x, t) dt \right\} dx \leq \int_{\mathbb{R}} \|g\|_q \|f(x, \cdot)\|_p dx \\ &= \|g\|_q \int_{\mathbb{R}} \|f(x, \cdot)\|_p dx. \end{aligned}$$

2) We may of course assume that $f(x, t) \geq 0$, because we can in general replace f by $|f|$, which gives a more “narrow” estimate. Then we can use the result from 1.

Let $p > 1$. Then

$$\begin{aligned} \|g\|_q &= \left\{ \int_{\mathbb{R}} \left(\int_{\mathbb{R}} f(x, t) dx \right)^{(p-1) \cdot \frac{p}{p-1}} dt \right\}^{\frac{p-1}{p}} = \left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}} f(x, t) dx \right)^p dt \right)^{\frac{1}{p}} \\ &= \left\| \int_{\mathbb{R}} f(x, \cdot) dx \right\|_p^{p-1}, \end{aligned}$$

which inserted into the result of 1) gives

$$\left\| \int_{\mathbb{R}} f(x, \cdot) dx \right\|_p^p \leq \left\| \int_{\mathbb{R}} f(x, \cdot) dx \right\|_o^{p-1} \cdot \int_{\mathbb{R}} \|f(x, \cdot)\|_p dx.$$

Since $p > 1$, this is reduced to

$$\left\| \int_{\mathbb{R}} f(x, \cdot) dx \right\|_p \leq \int_{\mathbb{R}} \|f(x, \cdot)\|_p dx.$$

When $p = 1$, then we get instead by interchanging the order of integration

$$\left\| \int_{\mathbb{R}} f(x, \cdot) dx \right\|_1 = \int_{\mathbb{R}} \left\{ \int_{\mathbb{R}} f(x, t) dx \right\} dt = \int_{\mathbb{R}} \left\{ \int_{\mathbb{R}} f(x, t) dt \right\} dx = \int_{\mathbb{R}} \|f(x, \cdot)\|_1 dt.$$

For a general f we get

$$\left\| \int_{\mathbb{R}} f(x, \cdot) dx \right\|_1 \leq \left\| \int_{\mathbb{R}} |f(x, \cdot)| dx \right\|_p \leq \int_{\mathbb{R}} \|f(x, \cdot)\|_p dx,$$

because $\| |f(x, \cdot)| \|_p = \|f(x, \cdot)\|_p$.

3) Given $h \in L^1(\mathbb{R})$ - Define an operator T by

$$Tg(x) = g \star h(x),$$

for the $g \in L^p(\mathbb{R})$, $p > 1$, for which this expression makes sense. Then clearly, T is linear.

Let $g \in L^p(\mathbb{R})$. Using 2) above we get the following estimate, where we allow ourselves to write $\|g \star h\|$ before we have proved that it makes sense,

$$\begin{aligned} \|Tg\|_p &= \|g \star h\|_p = \left\| \int_{\mathbb{R}} g(\star - x) h(x) dx \right\|_p \\ &\leq \int_{\mathbb{R}} \|g(\star - x)\|_p \cdot h(x) dx = \|g\|_p \cdot \|h\|_1 < \infty. \end{aligned}$$

This estimate shows that $g \star h \in L^p(\mathbb{R})$ is defined and that the mapping T is bounded of norm $\|T\| \leq \|h\|_1$.

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